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SCALES IN $K(\mathbb{R})$ AT THE END OF A WEAK GAP

J. R. STEEL

In this note we shall prove

THEOREM 0.1. *Let $\mathscr{M}$ be a countably ω-iterable $\mathbb{R}^{\mathscr{M}}$-mouse which satisfies* AD, *and $[\alpha, \beta]$ a weak gap of $\mathscr{M}$. Suppose $\Sigma_1^{\mathscr{M}|\alpha}$ is captured by mice with iteration strategies in $\mathscr{M}|\alpha$.*[1] *Let n be least such that $\rho_n(\mathscr{M}|\beta) = \mathbb{R}^{\mathscr{M}}$; then we have that $\mathscr{M}$ believes that $\Sigma_n^{\mathscr{M}|\beta}$ has the Scale Property.*

This complements the work of [5] on the construction of scales of minimal complexity on sets of reals in $K(\mathbb{R})$. Theorem 0.1 was proved there under the stronger hypothesis that all sets definable over $\mathscr{M}$ are determined, although without the capturing hypothesis. (See [5, Theorem 4.14].) Unfortunately, this is more determinacy than would be available as an induction hypothesis in a core model induction. The capturing hypothesis, on the other hand, is available in such a situation. Since core model inductions are one of the principal applications of the construction of optimal scales , it is important to prove 0.1 as stated.

Our proof will incorporate a number of ideas due to Woodin which figure prominently in the weak gap case of the core model induction. It relies also on the connection between scales and iteration strategies with the Dodd-Jensen property first discovered in [3]. Let $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})^{\mathscr{M}}}$ be the pointclass at the beginning of the weak gap referred to in 0.1. In section 1, we use Woodin's ideas to construct a Γ-*full* a mouse $\mathscr{Q}$ having ω Woodin cardinals cofinal in its ordinals, together with an iteration strategy Σ which *condenses well* in the sense of [4, Def. 1.13]. In section 2, we construct the desired scale from $\mathscr{Q}$ and Σ.

The reader should see sections 1 and 2 of [5] for an elementary discussion of $K(\mathbb{R})$. Sections 4.2 and 4.3 of [5] introduce the notion of a Σ_1-*gap* in $K(\mathbb{R})$, and use it to describe the pattern of scales in $K(\mathbb{R})$. We shall assume the reader is familiar with the definitions and statements of results in section 4 of [5]. It is not necessary to know any proofs there.

§1. A fullness-preserving iteration strategy. For the rest of this note we adopt the hypotheses of 0.1. We may as well also assume $\omega\beta = o(\mathscr{M})$. To make some smaller points easier to handle, we shall assume β is a limit ordinal, $\rho_1(\mathscr{M}) = \mathbb{R}$, and $\mathscr{M}$ is passive. (A little extra care is needed if $\mathscr{M}$ is active of type II.) We may also assume

$$\mathscr{M} \models \Theta \text{ exists},$$

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[1] This capturing hypothesis is explained below.

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