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MAXIMAL IRREDUNDANCE AND MAXIMAL IDEAL INDEPENDENCE IN BOOLEAN ALGEBRAS

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Introduction. Recall that a subset X of an algebra A is *irredundant* iff $x \notin \langle X\backslash\{x\}\rangle$ for all $x \in X$, where $\langle X\backslash\{x\}\rangle$ is the subalgebra generated by $X\backslash\{x\}$. By Zorn's lemma there is always a maximal irredundant set in an algebra. This gives rise to a natural cardinal function $\mathrm{Irr}_{\mathrm{mm}}(A) = \min\{|X| \colon X$ is a maximal irredundant subset of $A\}$. The first half of this article is devoted to proving that there is an atomless Boolean algebra A of size 2^ω for which $\mathrm{Irr}_{\mathrm{mm}}(A) = \omega$.

A subset X of a BA A is *ideal independent* iff $x \notin \langle X\backslash\{x\}\rangle^{\mathrm{id}}$ for all $x \in X$, where $\langle X\backslash\{x\}\rangle^{\mathrm{id}}$ is the ideal generated by $X\backslash\{x\}$. Again, by Zorn's lemma there is always a maximal ideal independent subset of any Boolean algebra. We then consider two associated functions. A spectrum function

$$\mathrm{s}_{\mathrm{spect}}(A) = \{|X| \colon X \text{ is a maximal ideal independent subset of } A\}$$

and the least element of this set, $\mathrm{s}_{\mathrm{mm}}(A)$. We show that many sets of infinite cardinals can appear as $\mathrm{s}_{\mathrm{spect}}(A)$. The relationship of s_{mm} to similar "continuum cardinals" is investigated. It is shown that it is relatively consistent that $\mathrm{s}_{\mathrm{mm}}(\mathfrak{P}(\omega)/\mathrm{fin}) < 2^\omega$.

We use the letter s here because of the relationship of ideal independence with the well-known cardinal invariant *spread*; see Monk [5]. Namely, $\sup\{|X| \colon X$ is ideal independent in $A\}$ is the same as the spread of the Stone space $\mathrm{Ult}(A)$; the spread of a topological space X is the supremum of cardinalities of discrete subspaces.

Notation. Our set-theoretical notation is standard, with some possible exceptions, as follows. limord is the class of all limit ordinals, and reg is the class of all regular cardinals. If α and β are ordinals, then $[\alpha, \beta]_{\mathrm{card}}$ is the collection of all cardinals κ such that $\alpha \le \kappa \le \beta$; similarly $[\alpha, \beta]_{\mathrm{reg}}$ for the collection of all regular cardinals in this interval; and similarly for other intervals (half open, rays, etc.).

We follow Koppelberg [2] for Boolean algebraic notation, and Monk [5] for more specialized notation concerning cardinal functions on BAs. $\mathrm{Fr}(\kappa)$ is the free BA on κ generators. $\overline{A}$ is the completion of A. In several places we use the following construction. Let $\langle A_i \colon i \in I\rangle$ be a system of BAs, with I infinite. The *weak product* $\prod^{\mathrm{w}}_{i\in I} A_i$ consists of all members x of the full product such that one of the two sets

$$\{i \in I \colon x_i \ne 0\} \quad \text{or} \quad \{i \in I \colon x_i \ne 1\}$$

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