THE JOURNAL OF SYMBOLIC LOGIC
Volume 72, Number 4, Dec. 2007

CLASSIFYING BOREL AUTOMORPHISMS

JOHN D. CLEMENS

§1. Introduction. This paper considers several complexity questions regarding *Borel automorphisms* of a Polish space. Recall that a Borel automorphism is a bijection of the space with itself whose graph is a Borel set (equivalently, the inverse image of any Borel set is Borel). Since the inverse of a Borel automorphism is another Borel automorphism, as is the composition of two Borel automorphisms, the set of Borel automorphisms of a given Polish space forms a group under the operation of composition. We can also consider the class of automorphisms of all Polish spaces. We will be primarily concerned here with the following notion of equivalence:

DEFINITION 1.1. Two Borel automorphisms f and g of the Polish spaces X and Y are said to be Borel isomorphic, $f \cong g$, if they are conjugate, i.e. there is a Borel bijection $\varphi : X \to Y$ such that $\varphi \circ f = g \circ \varphi$.

We restrict ourselves to automorphisms of uncountable Polish spaces, as the Borel automorphisms of a countable space are simply the permutations of the space. Since any two uncountable Polish spaces are Borel isomorphic, any Borel automorphism is Borel isomorphic to some automorphism of a fixed space. Hence, up to Borel isomorphism we can fix a Polish space and represent any Borel automorphism as an automorphism of this space. We will use the Cantor space 2^ω (with the product topology) as our representative space.

We may then represent a Borel automorphism by its graph, which is a subset of $(2^\omega)^2$. This graph is a Borel set, and may thus be coded as a real using a coding of Borel sets. The set of Borel automorphisms can then be viewed as a set of reals, and the relation of Borel isomorphism as an equivalence relation on this set. This allows us to analyze the complexity of this relation using descriptive set-theoretic techniques. Two natural questions arise:

1. How complicated is this equivalence relation descriptively; i.e., where does it fall in the Wadge hierarchy?
2. How complicated is this relation in the hierarchy of equivalence relations under Borel reducibility?

We will be able to completely answer the first question by showing that the isomorphism relation is $\mathbf{\Sigma}^1_2$-complete. We will be able to give a partial answer to the second question by showing that the relation is quite complicated: The equivalence relation

Received November 29, 2002.