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SOME RESULTS IN POLYCHROMATIC RAMSEY THEORY

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§1. Introduction. *Classical Ramsey theory* (at least in its simplest form) is concerned with problems of the following kind: given a set X and a colouring of the set $[X]^n$ of unordered n-tuples from X, find a subset $Y \subseteq X$ such that all elements of $[Y]^n$ get the same colour. Subsets with this property are called *monochromatic* or *homogeneous*, and a typical positive result in Ramsey theory has the form that when X is large enough and the number of colours is small enough we can expect to find reasonably large monochromatic sets.

Polychromatic Ramsey theory is concerned with a "dual" problem, in which we are given a colouring of $[X]^n$ and are looking for subsets $Y \subseteq X$ such that any two distinct elements of $[Y]^n$ get *different* colours. Subsets with this property are called *polychromatic* or *rainbow*. Naturally if we are looking for rainbow subsets then our task becomes easier when there are many colours. In particular given an integer k we say that a colouring is k-*bounded* when each colour is used for at most k many n-tuples.

At this point it will be convenient to introduce a compact notation for stating results in polychromatic Ramsey theory. We recall that in classical Ramsey theory we write $\kappa \to (\alpha)^n_k$ to mean "every colouring of $[\kappa]^n$ in k colours has a monochromatic set of order type α". We will write $\kappa \to^{\text{poly}} (\alpha)^n_{k\text{-bd}}$ to mean "every k-bounded colouring of $[\kappa]^n$ has a polychromatic set of order type α". We note that when κ is infinite and k is finite a k-bounded colouring will use exactly κ colours, so we may as well assume that κ is the set of colours used.

Polychromatic Ramsey theory in the finite case has been extensively studied by finite combinatorists [3, 7, 10, 12], sometimes under the name "Rainbow Ramsey theory" or "Sub-Ramsey theory". In particular the quantity $\mathrm{sr}(K_n, k)$, which in our notation is the least m such that $m \to^{\text{poly}} (n)^2_{k\text{-bd}}$, has been investigated; it grows *much* more slowly than the corresponding classical Ramsey number, for example it is known [3] that $\mathrm{sr}(K_n, k) \leq \frac{1}{4}n(n-1)(n-2)(k-1) + 3$.

In this paper we investigate polychromatic versions of some classic results in infinite Ramsey theory. Interestingly we will also often find that, as in the finite case, polychromatic Ramsey numbers grow more slowly than their classical counterparts.

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