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LOCAL K^c CONSTRUCTIONS

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The full-background-extender K^c-construction of [2] has the property that, if it does not break down and produces a final model $L[\vec{E}]$, then

$$\delta \text{ is Woodin in } V \Rightarrow \delta \text{ is Woodin in } L[\vec{E}],$$

for all δ. It is natural to ask whether

$$\kappa \text{ is strong in } V \Rightarrow \kappa \text{ is strong in } L[\vec{E}],$$

for all κ, or even better,

$$\kappa \text{ is } \lambda\text{-strong in } V \Rightarrow \kappa \text{ is } \lambda\text{-strong in } L[\vec{E}].$$

As one might suspect, the more useful answer would be "yes".

For the K^c-construction of [2], this question is open. The problem is that the construction of [2] is not local: because of the full-background-extender demand, it may produce mice projecting to ρ at stages much greater than ρ. Because of this, there is no reason to believe that if E is a λ-strong extender of V, then $i_E(\vec{E}) \restriction \lambda = \vec{E} \restriction \lambda$. The natural proof only gives that if κ is Σ_2-strong, then κ is strong in $L[\vec{E}]$.[1]

We do not know how to get started on this question, and suspect that in fact strong cardinals in V may fail to be strong in $L[\vec{E}]$, if $L[\vec{E}]$ is the output of the construction of [2]. Therefore, we shall look for a modification of the construction of [2]. One might ask for a construction with output $L[\vec{E}]$ such that

(1) iteration trees on $L[\vec{E}]$ can be lifted to iteration trees on V,
(2) $\forall\delta(\delta$ is Woodin $\Rightarrow \delta$ is Woodin in $L[\vec{E}])$, and
(3) (a) $\forall\kappa(\kappa$ is a strong cardinal $\Rightarrow \kappa$ is strong in $L[\vec{E}])$, and
    (b) $\forall\kappa\forall\lambda(\text{Lim}(\lambda) \wedge \kappa$ is λ-strong $\Rightarrow \kappa$ is λ-strong in $L[\vec{E}])$.

In §1 we shall describe a construction which satisfies these demands, but we shall need to assume that V itself is a fine-structural inner model $L[\vec{F}][x]$, built over some set x.[2] In this case, the iterability reduction of (1) will produce iteration trees on V which may drop, though only when the tree on $L[\vec{E}]$ being lifted also drops. A reduction to non-dropping trees would have some use. We believe we

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[1]This question, and the observations we have just recorded, are due to R. Jensen.

[2]There are applications of this result in [6]. For example, the minimal model of the $\mathsf{AD}_{\mathbb{R}}$-hypothesis, and related mice, reconstruct themselves below the sup of their Woodins via our modified construction, and this helps analyze the derived models of such mice. See [6, §3, §5].

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