A DICHOTOMY THEOREM FOR TURBULENCE

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§1. Introduction. In this note we show:

THEOREM 1.1. *Let G be a Polish group and X a Polish G-space with the induced orbit equivalence relation E_G Borel as a subset of $X \times X$. Then exactly one of the following*:

(I) *There is a countable language $\mathscr{L}$ and a Borel function*

$$\theta \colon X \to \mathrm{Mod}(\mathscr{L})$$

such that for all $x_1, x_2 \in X$

$$x_1 \, E_G \, x_2 \iff \theta(x_1) \cong \theta(x_2);$$

or

(II) *there is a turbulent Polish G-space Y and a continuous G-embedding*

$$\tau \colon Y \hookrightarrow X.$$

There are various bows and ribbons which can be woven into these statements. We can strengthen (I) by asking that θ also admit a Borel orbit inverse, that is to say some Borel function

$$\rho \colon B \to X,$$

for some Borel set $B \subset \mathrm{Mod}(\mathscr{L})$, such that for all $x \in X$

$$x \, E_G \, \rho(\theta(x));$$

and then after having passed to this strengthened version of (I) we still obtain the exact same dichotomy theorem, and hence the conclusion that the two competing versions of (I) are equivalent. Similarly (II) can be relaxed to just asking that τ be a Borel G-embedding, or even simply a Borel reduction of the relevant orbit equivalence relations. It is in fact a consequence of 1.1 that all the plausible weakenings and strengthenings of (I) and (II) are respectively equivalent to one another.

I will not closely examine these possible variations here. The equivalences alluded to above follow from our main theorem and the results of [3]. That monograph had previously shown that (I) and (II) are incompatible, and proved a barbaric forerunner of 1.1, and gone on to conjecture the dichotomy result above.