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A CONSTRUCTIVE LOOK AT THE COMPLETENESS OF THE SPACE $\mathscr{D}(\mathbb{R})$

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Abstract. We show, within the framework of Bishop's constructive mathematics, that (sequential) completeness of the locally convex space $\mathscr{D}(\mathbb{R})$ of test functions is equivalent to the principle BD-$\mathbb{N}$ which holds in classical mathemtatics, Brouwer's intuitionism and Markov's constructive recursive mathematics, but does not hold in Bishop's constructivism.

§1. Introduction. The space $\mathscr{D}(\mathbb{R})$ of all infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with compact support together with a locally convex structure defined by the seminorms

$$p_{\alpha,\beta}(f) := \sup_n \max_{l \leq \beta(n)} \sup_{|x| \geq n} 2^{\alpha(n)} |f^{(l)}(x)| \quad (\alpha, \beta \in \mathbb{N} \to \mathbb{N})$$

is an important example of a locally convex space. Classically the space $\mathscr{D}(\mathbb{R})$—the space of test functions—is complete, but it has not been known whether the constructive completion of $\mathscr{D}(\mathbb{R})$, whose explicit description was given in [1, Appendix A] and [2, Chapter 7, Notes], coincides with the original space or not. This leads us to a difficulty in developing the theory of distributions in Bishop's constructive mathematics; see [1, Appendix A] and [2, Chapter 7, Notes] for more details.

The aim of our paper is to find a principle which is necessary and sufficient to establish the completeness of $\mathscr{D}(\mathbb{R})$. Although it is formulated in the setting of informal Bishop-style constructive mathematics, the proofs could easily be formalized in a system based on intuitionistic finite-type arithmetics $\boldsymbol{HA}^{\omega}$ [8, Chapter 1], [9, Chapter 9]; see also [5].

A subset A of $\mathbb{N}$ is said to be *pseudobounded* if for each sequence $\{a_n\}_n$ in A,

$$\lim_{n\to\infty} \frac{a_n}{n} = 0.$$

A bounded subset of $\mathbb{N}$ is pseudobounded. The converse for countable sets holds in in classical mathematics, intuitionistic mathematics and constructive recursive mathematics of Markov's school; see [6]. However, a natural recursivisation of the following principle is independent of Heyting arithmetic [4].

BD-$\mathbb{N}$**:** Every countable pseudobounded subset of $\mathbb{N}$ is bounded.

BD-$\mathbb{N}$ has been proved to be equivalent to the following theorems [6, 7, 4]; Banach's inverse mapping theorem; the open mapping theorem; the closed graph theorem;

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