THE JOURNAL OF SYMBOLIC LOGIC
Volume 67, Number 4, Dec. 2002

ON ITERATING SEMIPROPER PREORDERS

TADATOSHI MIYAMOTO

Abstract. Let T be an ω_1-Souslin tree. We show the property of forcing notions ; "is $\{\omega_1\}$-semi-proper and preserves T" is preserved by a new kind of revised countable support iteration of arbitrary length. As an application we have a forcing axiom which is compatible with the existence of an ω_1-Souslin tree for preorders as wide as possible.

Introduction. We develop a way to take a limit in iterated forcing (lemma (2.9)). The limit allows us to deal with more conditions than countable support iterations (corollary (3.3)), but we still have a reasonable control over the conditions (proposition (3.9)). We show that the property of forcing notions; "is $\{\omega_1\}$-semi-proper" is preserved by the iterations of arbitrary length which take this new limit at every limit stage (lemma (4.2)). Note that this assumes no cardinal collapses at all (cf. corollary 2.8 on p. 320 in [7]).

We also show, if T is an ω_1-Souslin tree, then the property of forcing notions;"is $\{\omega_1\}$-semi-proper and preserves T" is preserved by this type of iteration of arbitrary length (theorem (5.1)). This continues [5] in which we dealt with proper preorders and countable support iterations. As an application we get a forcing axiom which is compatible with the existence of an ω_1-Souslin tree for the preorders which preserve every stationary subset of ω_1 and preserve every ω_1-Souslin tree assuming the existence of a supercompact cardinal (corollary (5.8)).

While our note, particularly §5, depends on [1], [2] and [7] in many respects, we give most proofs in detail for the sake of completeness. In §1, we prepare a basic theory of iterated forcing. In §2, we introduce our main definitions involved in defining our limit. In §3, we develop the theory of nice iterations. In §4, we show the iteration theorem for $\{\omega_1\}$-semi-proper. In §5, we show a couple of iteration theorems along the same line and deal with forcing axioms. In §6, we briefly look at other approaches to revised countable support iterations hinted on [3], [6] and the second edition [9] of [7].

§0. Preliminary. In this note we deal with *separative preorders* $(P, \leq, 1)$. By this we mean that $\leq$ is a reflexive (i.e., for all $p \in P$, $p \leq p$) and transitive (i.e., for all $p, q, r \in P$, if $r \leq q$ and $q \leq p$, then $r \leq p$) binary relation on P with a greatest element 1 (i.e., for all $p \in P$, $p \leq 1$) and that for any $p, q \in P$, $q \leq p$ iff $q \Vdash_P$"$p \in \dot{G}$", where $\dot{G}$ denotes the canonical P-name of the P-generic filters

Received July 21, 1993; revised April 4, 2002.

0022-4812/02/6704-0011/$4.80