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SMALL THEORIES OF BOOLEAN ORDERED O-MINIMAL STRUCTURES

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Abstract. We investigate small theories of Boolean ordered o-minimal structures. We prove that such theories are $\aleph_0$-categorical. We give a complete characterization of their models up to bi-interpretability of the language. We investigate types over finite sets, formulas and the notions of definable and algebraic closure.

§0. Introduction. The notion of o-minimality for partially ordered structures is a natural generalization of o-minimality for totally ordered structures, defined in [4]. We say that the partially ordered structure $(M, \leq, \dots)$ is quasi o-minimal iff every definable set $X \subseteq M$ is a Boolean combination of sets defined by formulas $x \leq a$, $x \geq b$ with $a, b \in M$. If, additionally, the parameters a and b can be chosen algebraic over the parameters of the formula defining X, we say that $(M, \leq, \dots)$ is o-minimal. This distinction was drawn by C. Toffalori in [5]. In this paper we deal with Boolean ordered structures, i.e., partially ordered structures where the ordering is that of a Boolean algebra. Since, as shown in [3], the notion of o-minimality in Boolean ordered structures coincides with that of quasi o-minimality, it is never necessary to distinguish between them, and we shall never again do so.

The paper is organized as follows. In §1 we recall some useful notions and facts from [3]. §2 contains the main result of this paper, namely that a small theory of a Boolean ordered o-minimal structure is $\aleph_0$-categorical. In §3 we present some properties of the operators *dcl* and *acl* in Boolean ordered o-minimal structures with small theory. We find a canonical form of formulas, which enables us to classify all Boolean ordered o-minimal structures with a small theory up to bi-interpretability of the language. In our setting we find an analogue of binarity, a property which is enjoyed by small theories of totally ordered o-minimal structures.

§1. Notation and preliminaries. Let $(M, \sqcap, \sqcup, ', 0_M, 1_M)$ be an arbitrary Boolean algebra and $a \in M \setminus \{0_M\}$. For $x \in M$ we define x'^a as $x' \sqcap a$. Then $([0_M, a], \sqcap, \sqcup, '^a, 0_M, a)$ is a Boolean algebra. $[0_M, a]$ is a standard (closed) interval in M with endpoints 0_M and a. For a non-empty set $B \subseteq M$ we define

$$U(a, B) = \{x \sqcap a : x \text{ is a Boolean combination of elements from } B\}.$$

$(U(a, B), \sqcap, \sqcup, '^a, 0_M, a)$ is again a Boolean algebra.

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