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A UNIQUENESS THEOREM FOR ITERATIONS

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Abstract. If M is a countable transitive model of ZFC+$\mathrm{MA}_{\aleph_1}$, then for every real x there is a unique shortest iteration $j\colon M \to N$ with $x \in N$, or none at all.

The fundamental construction underlying Woodin's $\mathbb{P}_{max}$ forcing [4] is the iterated generic elementary embedding. In this construction, one takes a countable transitive model M of ZFC, chooses an M-generic filter $G \subset (\mathscr{P}(\omega_1)/I_{NS})^M$ (where I_{NS} denotes the nonstationary ideal) and constructs the corresponding ultrapower of M, with its associated embedding. The process is then repeated with the ultrapower, and continued for ω_1 many stages, taking direct limits at limit stages. The resulting chain of models, with their corresponding embeddings, is called an iteration of M, and a great deal of the $\mathbb{P}_{max}$ analysis concerns the study of these iterations.

A well known result, which appears in [4] as Theorem 5.101 and here as Corollary 3.3, says that two different generic ultrapowers of the same model of ZFC + $\mathrm{MA}_{\aleph_1}$ have no new reals in common. In this paper we strengthen this result to iterations of arbitrary countable length. Our approach gives a shorter proof of the original result as well.

A more precise statement of our result uses the following cardinal invariant.

DEFINITION 0.1. The cardinal invariant $\mathfrak{q}$ is defined to be the least κ such that there is an almost disjoint family $\{x_\alpha : \alpha < \kappa\}$ of subsets of ω and a set $A \subset \kappa$ such that for no $z \subset \omega$ is it true that for all $\alpha < \kappa$, $\alpha \in A$ if and only if $z \cap x_\alpha$ is infinite.

It is a standard consequence of $MA_{\aleph_1}$ that $\mathfrak{q} > \omega_1$ (see, for example, [1]).

THEOREM 0.2. *Let M be a countable iterable model of ZFC + $\mathfrak{q} > \aleph_1$, and let $j_0\colon M \to N_0$ and $j_1\colon M \to N_1$ be iterations of M whose first steps are distinct. Then* $\mathbb{R}^{N_0} \cap \mathbb{R}^{N_1} = \mathbb{R}^M$.

It is hoped that a better understanding of iterations will lead to a finer analysis of $\mathbb{P}_{max}$ variations and their extensions. Other problems in this class appear in [2, 3].

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