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SOME HIGHLY SATURATED MODELS OF PEANO ARITHMETIC

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Some highly saturated models of Peano Arithmetic are constructed in this paper, which consists of two independent sections. In §1 we answer a question raised in [10] by constructing some highly saturated, rather classless models of PA. A question raised in [7], [3], [4] is answered in §2, where highly saturated, nonstandard universes having no bad cuts are constructed.

Highly saturated, rather classless models of Peano Arithmetic were constructed in [10]. The main result proved there is the following theorem. If λ is a regular cardinal and $\mathscr{M}$ is a λ-saturated model of PA such that $|M| > \lambda$, then $\mathscr{M}$ has an elementary extension $\mathscr{N}$ of the same cardinality which is also λ-saturated and which, in addition, is rather classless. The construction in [10] produced a model $\mathscr{N}$ for which $\mathrm{cf}(\mathscr{N}) = \lambda^+$. We asked in Question 5.1 of [10] what other cofinalities could such a model $\mathscr{N}$ have. This question is answered here in Theorem 1.1 of §1 by showing that any cofinality not immediately excluded is possible. Its proof does not depend on the theorem from [10]; in fact, the proof presented here gives a proof of that theorem which is much simpler and shorter than the one in [10].

Recursively saturated, rather classless κ-like models of PA were constructed in [9]. In the case of singular κ such models were constructed whenever $\mathrm{cf}(\kappa) > \aleph_0$; no additional set-theoretic hypothesis was needed. The construction in the proof of Theorem 1.1 can be modified to obtain highly saturated, rather classless κ-like models whenever $\mathrm{cf}(\kappa) > \aleph_0$. This result appears as Theorem 1.2 of §1.

Shelah [11] proved that every nonstandard universe and, in fact, every nonstandard model of PA has a (λ, λ)-cut for some cardinal λ. Jin [4] proved the stronger result that every nonstandard model of PA has type two cuts. That left as unsettled the possibility of an even stronger result, that every nonstandard universe has a bad cut. (Bad cuts will be defined in §2. Bad cuts and type two cuts were introduced in [7], which should be consulted for the definition of type two cuts and for the relevance of both bad and type two cuts.) It will be shown in §2 that it is consistent that there are highly saturated nonstandard universes having no bad cuts. We find it more convenient to work with models of PA in §2. A consequence of the main result of that section is that, assuming $\diamondsuit_{\lambda^+}$ for some uncountable, regular λ, there are λ-saturated models of PA having no bad cuts. The proof of this can easily be applied to nonstandard universes.

Consult Kaye [6] as a basic reference to models of PA. It is possible to read either of §§1 or 2 without reading the other one.

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