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# SEQUENCES OF $n$-DIAGRAMS

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**§1. Introduction.** We consider only computable languages, and countable structures, with universe a subset of $\omega$, which we think of as a set of constants. We identify sentences with their Gödel numbers. Thus, for a structure $\mathscr{A}$, the complete (elementary) diagram, $D^c(\mathscr{A})$, and the atomic diagram, $D(\mathscr{A})$, are subsets of $\omega$. We classify formulas as usual. A formula is both $\Sigma_0$ and $\Pi_0$ if it is open. For $n > 0$, a formula, in prenex normal form, is $\Sigma_n$, or $\Pi_n$, if it has $n$ blocks of like quantifiers, beginning with $\exists$, or $\forall$. For a formula $\theta$, in prenex normal form, we let $neg(\theta)$ denote the dual formula that is logically equivalent to $\neg\theta$—if $\theta$ is $\Sigma_n$, then $neg(\theta)$ is $\Pi_n$, and vice versa.

DEFINITION 1.1. *For a structure $\mathscr{A}$, the* $n$-diagram *is*

$$D_n(\mathscr{A}) = D^c(\mathscr{A}) \cap \Sigma_n.$$

We are interested in complexity, which we measure by Turing degree. We denote Turing reducibility by $\leq_T$, and Turing equivalence by $\equiv_T$. It is clear that for any structure $\mathscr{A}$, $D_0(\mathscr{A}) \equiv_T D(\mathscr{A})$. We show that for any $\mathscr{A}$, there exists $\mathscr{B} \cong \mathscr{A}$ such that $D^c(\mathscr{B}) \equiv_T D(\mathscr{B})$. If $\mathscr{A}$ is an algebraically closed field, a real closed field, or any other structure in which we have effective elimination of quantifiers, then this collapse is "intrinsic"; i.e., it happens in all copies. For models of $PA$, the collapse is not intrinsic. For the standard model of arithmetic, $\mathscr{N} = (\omega, +, \cdot, S, 0)$, we have $D_n(\mathscr{N}) \equiv_T \emptyset^{(n)}$, uniformly in $n$. In [8], it is shown that for any model $\mathscr{A}$ of $PA$, there exists $\mathscr{B} \cong \mathscr{A}$ such that $D_{n+1}(\mathscr{B}) \not\leq_T D_n(\mathscr{B})$.

We first consider the following problem.

PROBLEM 1. *Find syntactic conditions on $\mathscr{A}$ guaranteeing that for some $n$, for all $\mathscr{B} \cong \mathscr{A}$, $D^c(\mathscr{B}) \equiv_T D_n(\mathscr{B})$. In particular, for $n = 0$, find syntactic conditions guaranteeing that for all $\mathscr{B} \cong \mathscr{A}$, $D^c(\mathscr{B}) \equiv_T D(\mathscr{B})$.*

For structures $\mathscr{A}$ that do not exhibit intrinsic collapse of the complete diagram to the atomic diagram, we consider the sequences $(D_n(\mathscr{B}))_{n\in\omega}$ for $\mathscr{B} \cong \mathscr{A}$. We focus on the corresponding sequences of Turing degrees.

DEFINITION 1.2. (i) *For sets $X$ and $Y$, $Y$ is* c.e. in and above *$X$ if $Y$ is c.e. relative to $X$, and $X \leq_T Y$.*

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