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ON REGULAR REDUCED PRODUCTS*

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Abstract. Assume $\langle\aleph_0, \aleph_1\rangle \rightarrow \langle\lambda, \lambda^+\rangle$. Assume M is a model of a first order theory T of cardinality at most λ^+ in a language $\mathscr{L}(T)$ of cardinality $\leq \lambda$. Let N be a model with the same language. Let Δ be a set of first order formulas in $\mathscr{L}(T)$ and let D be a regular filter on λ. Then M is Δ-embeddable into the reduced power N^λ/D, provided that every Δ-existential formula true in M is true also in N. We obtain the following corollary: for M as above and D a regular ultrafilter over λ, M^λ/D is λ^{++}-universal. Our second result is as follows: For $i < \mu$ let M_i and N_i be elementarily equivalent models of a language which has cardinality $\leq \lambda$. Suppose D is a regular filter on λ and $\langle\aleph_0, \aleph_1\rangle \rightarrow \langle\lambda, \lambda^+\rangle$ holds. We show that then the second player has a winning strategy in the Ehrenfeucht-Fraïssé game of length λ^+ on $\prod_i M_i/D$ and $\prod_i N_i/D$. This yields the following corollary: Assume GCH and λ regular (or just $\langle\aleph_0, \aleph_1\rangle \rightarrow \langle\lambda, \lambda^+\rangle$ and $2^\lambda = \lambda^+$). For L, M_i and N_i be as above, if D is a regular filter on λ, then $\prod_i M_i/D \cong \prod_i N_i/D$.

§1. Introduction. Suppose M is a first order structure and F is the Frechet filter on ω. Then the reduced power M^ω/F is $\aleph_1$-saturated and hence $\aleph_2$-universal ([6]). This was generalized by Shelah in [10] to any filter F on ω for which B^ω/F is $\aleph_1$-saturated, where B is the two element Boolean algebra, and in [8] to all regular filters on ω. In the first part of this paper we use the combinatorial principle $\square^{b^*}_\lambda$ of Shelah [11] to generalize the result from ω to arbitrary λ, assuming $\langle\aleph_0, \aleph_1\rangle \rightarrow \langle\lambda, \lambda^+\rangle$. This gives a partial solution to Conjecture 19 in [3]: if D is a regular ultrafilter over λ, then for all infinite M, the ultrapower M^λ/D is λ^{++}-universal.

The second part of this paper addresses Problem 18 in [3], which asks if it is true that if D is a regular ultrafilter over λ, then for all elementarily equivalent models M and N of cardinality $\leq \lambda$ in a language of cardinality $\leq \lambda$, the ultrapowers M^λ/D and N^λ/D are isomorphic. Keisler [7] proved this for good D assuming $2^\lambda = \lambda^+$. Benda [1] weakened "good" to "contains a good filter". We prove the claim in full generality, assuming $2^\lambda = \lambda^+$ and $\langle\aleph_0, \aleph_1\rangle \rightarrow \langle\lambda, \lambda^+\rangle$.

Regarding our assumption $\langle\aleph_0, \aleph_1\rangle \rightarrow \langle\lambda, \lambda^+\rangle$, by Chang's Two-Cardinal Theorem ([2]) $\langle\aleph_0, \aleph_1\rangle \rightarrow \langle\lambda, \lambda^+\rangle$ is a consequence of $\lambda = \lambda^{<\lambda}$. So our Theorem 2 settles Conjecture 19 of [3], and Theorem 13 settles Conjecture 18 of [3], under GCH for λ regular. For singular strong limit cardinals $\langle\aleph_0, \aleph_1\rangle \rightarrow \langle\lambda, \lambda^+\rangle$ follows from $\square_\lambda$

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