THE JOURNAL OF SYMBOLIC LOGIC
Volume 67, Number 3, Sept. 2002

BOUNDED MARTIN'S MAXIMUM, WEAK ERDŐS CARDINALS, AND ψ_{AC}

DAVID ASPERÓ AND PHILIP D. WELCH*

Abstract. We prove that a form of the Erdős property (consistent with $V = L[H_{\omega_2}]$ and strictly weaker than the Weak Chang's Conjecture at ω_1), together with Bounded Martin's Maximum implies that Woodin's principle ψ_{AC} holds, and therefore $2^{\aleph_0} = \aleph_2$. We also prove that ψ_{AC} implies that every function $f: \omega_1 \to \omega_1$ is bounded by some canonical function on a club and use this to produce a model of the Bounded Semiproper Forcing Axiom in which Bounded Martin's Maximum fails.

§1. Introduction. Recall the following bounded form of Martin's Maximum ([Fo-M-S]), the maximal forcing axiom for collections of $\aleph_1$-many antichains:

DEFINITION 1.1. *Bounded Martin's Maximum* (*BMM*) *is the following statement*: *Suppose* $\mathbb{P}$ *is a stationary-set-preserving poset* (*i.e., every stationary subset of* ω_1 *remains stationary after forcing with* $\mathbb{P}$) *and* $\langle A_i : i < \omega_1\rangle$ *is a sequence of maximal antichains of* $\mathbb{P}$ *of size at most* $\aleph_1$. *Then there is a filter* $G \subseteq \mathbb{P}$ *such that* $G \cap A_i \neq \emptyset$ *for all* $i < \omega_1$.

Bounded forcing axioms, and *BMM* in particular, can be characterized as principles of generic absoluteness for Σ_1 formulas with parameters in H_{ω_2}. More precisely, the following holds ([B]):

CHARACTERIZATION 1.1. *BMM holds if and only if for every* $a \in H_{\omega_2}$ *and every* Σ_1 *formula* $\varphi(x)$, $H_{\omega_2} \models \varphi(a)$ *iff there is some stationary-set-preserving poset* $\mathbb{P}$ *such that* $\Vdash_{\mathbb{P}} \varphi(\check{a})$.

We shall also consider the bounded forcing axiom obtained from replacing "stationary-set-preserving" by "semiproper" in the Definition 1.1. This is called *Bounded Semiproper Forcing Axiom* (*BSPFA*). *BSPFA* can be characterized as a principle of generic absoluteness in a similar way as *BMM*. Specifically, in Characterization 1.1 one can replace "*BMM*" and "stationary-set-preserving" by "*BSPFA*" and "semiproper", respectively ([B]).

It turns out that *BSPFA* is equiconsistent with the existence of a so-called Σ_2-*reflecting cardinal*, i.e., an inaccessible cardinal κ such that $V_\kappa \prec_{\Sigma_2} V$, which is a

Received April 6, 2001; revised December 20, 2001.
1991 *Mathematics Subject Classification.* 03E02, 03E50, 03E55.
Key words and phrases. Bounded Martin's Maximum, ψ_{AC}, bounding by canonical functions, Erdős cardinals.
*Welch would like to thank the Institut für Formale Logik, Wien for a Gastprofessur during 2000-2002, where his part of this work was done.

0022-4812/02/6703-0018/$2.20