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STRONG CONVERGENCE IN FINITE MODEL THEORY

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Abstract. In [9] we introduced a new framework for asymptotic probabilities, in which a σ-additive measure is defined on the sample space of all sequences $\boldsymbol{A} =< \mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3, .. >$ of finite models, where the universe of $\mathscr{A}_n$ is $\{1, 2, .., n\}$. In this framework we investigated the strong 0-1 law for sentences, which states that each sentence either holds in $\mathscr{A}_n$ eventually almost surely or fails in $\mathscr{A}_n$ eventually almost surely.

In this paper we define the strong convergence law for formulas, which carries over the ideas of the strong 0-1 law to formulas with free variables, and roughly states that for each formula $\phi(\boldsymbol{x})$, the fraction of tuples $\boldsymbol{a}$ in $\mathscr{A}_n$, which satisfy the formula $\phi(\boldsymbol{x})$, almost surely has a limit as n tends to infinity.

We show that the infinitary logic with finitely many variables has the strong convergence law for formulas for the uniform measure, and further characterize the measures on random graphs for which the strong convergence law holds.

§1. Introduction. The 0-1 law for a logic $\mathscr{L}$ states that for each sentence ϕ in $\mathscr{L}$, the fraction $\mu_n(\phi)$ of the relational models with universe $\{1, 2, .., n\}$ satisfying ϕ has a limiting value 0 or 1 as n tends to infinity. The 0-1 law holds for first order logic [3, 2], and was further extended in several ways, see e.g., [1, 8].

It is known that first order logic does not admit a 0-1 law if the vocabulary is not purely relational. However, if the relational vocabulary v is extended with constant symbols $\boldsymbol{c}$, it follows from [3] that a convergence law holds, i.e., $\lim_n \mu_n(\phi(\boldsymbol{c}))$, where μ_n is the uniform measure on the set of all $(v \cup \boldsymbol{c})$-models with universe $\{1, 2, .., n\}$.

This paper strengthens this convergence law by isolating the convergence part due to the presence of $\boldsymbol{c}$. The stronger law will imply that for each formula $\phi(\boldsymbol{x})$, the fraction of tuples $\boldsymbol{c}$ satisfying $\phi(\boldsymbol{x})$ in almost all v-models is close to some value $r \in [0, 1]$.

This strong convergence law is naturally defined in a new framework, first introduced in [9], where a (σ-additive) measure is defined on the set of all increasing sequences of finite models.

Intuitively, we perform the experiment of randomly choosing models $\mathscr{A}_n$ with universe $\{1, .., n\}$, where the $\mathscr{A}_n$ are chosen independently of each other according to the measures μ_n.

In this framework the strong convergence law roughly states that for each formula $\phi(\boldsymbol{x})$, the fraction of tuples $\boldsymbol{a}$ in $\mathscr{A}_n$, which satisfy the formula $\phi(\boldsymbol{x})$, almost surely has a limit as n tends to infinity.

If $\boldsymbol{x}$ is empty, the fraction mentioned above is either 0 or 1, and the strong convergence law for formulas simply reduces to the strong 0-1 law for sentences [9],

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