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ISOLATION AND LATTICE EMBEDDINGS

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Abstract. Say that $(\boldsymbol{a}, \boldsymbol{d})$ is an isolation pair if $\boldsymbol{a}$ is a c.e. degree, $\boldsymbol{d}$ is a d.c.e. degree, $\boldsymbol{a} < \boldsymbol{d}$ and $\boldsymbol{a}$ bounds all c.e. degrees below $\boldsymbol{d}$. We prove that there are an isolation pair $(\boldsymbol{a}, \boldsymbol{d})$ and a c.e. degree $\boldsymbol{c}$ such that $\boldsymbol{c}$ is incomparable with $\boldsymbol{a}, \boldsymbol{d}$, and $\boldsymbol{c}$ cups $\boldsymbol{d}$ to $\mathbf{o}'$, caps $\boldsymbol{a}$ to $\mathbf{o}$. Thus, $\{\mathbf{o}, \boldsymbol{c}, \boldsymbol{d}, \mathbf{o}'\}$ is a diamond embedding, which was first proved by Downey in [9]. Furthermore, combined with Harrington-Soare continuity of capping degrees, our result gives an alternative proof of N_5 embedding.

§**1. Introduction.** A set $A \subseteq \omega$ is computably enumerable (c.e. for short), if A can be listed effectively. Say that $D \subseteq \omega$ is d.c.e. if D is the difference of two c.e. sets. A Turing degree is c.e. (d.c.e.) if it contains a c.e. (d.c.e.) set. Let $\boldsymbol{R}$ be the set of all c.e. degrees and $\boldsymbol{D}_2$ be the set of all d.c.e. degrees. Since any c.e. set is d.c.e., $\boldsymbol{R} \subseteq \boldsymbol{D}_2$. Say that a degree $\boldsymbol{d}$ is properly d.c.e. if $\boldsymbol{d}$ contains a d.c.e. set, but contains no c.e. sets. Cooper [3] proved the existence of properly d.c.e. degrees. Thus,

THEOREM 1 (Cooper [3]). $\boldsymbol{R} \subset \boldsymbol{D}_2$.

In [5], Cooper, Lempp and Watson proved that the properly d.c.e. degrees are dense in the c.e. degrees. The early investigation of the structure $\boldsymbol{D}_2$ shows that $\boldsymbol{D}_2$ shares many properties with $\boldsymbol{R}$. For example, Lachlan noticed that any nonzero d.c.e. degree bounds a nonzero c.e. degree, and so the downwards density holds in $\boldsymbol{D}_2$. Based on this observation, Jockusch pointed out that $\boldsymbol{D}_2$ is not complemented. The first two structural differences between $\boldsymbol{D}_2$ and $\boldsymbol{R}$ are obtained by Arslanov and Downey:

THEOREM 2 (Arslanov's Cupping Theorem [1]). *Every nonzero d.c.e. degree cups to* $\mathbf{o}'$ *with an incomplete d.c.e. degree.*

THEOREM 3 (Downey's Diamond Embedding Theorem [9]). *There are two d.c.e. degrees* $\boldsymbol{d}_1, \boldsymbol{d}_2$ *such that* $\boldsymbol{d}_1$ *cups* $\boldsymbol{d}_2$ *to* $\mathbf{o}'$ *and caps* $\boldsymbol{d}_2$ *to* $\mathbf{o}$.

In [8], Ding and Qian proved that one of $\boldsymbol{d}_1, \boldsymbol{d}_2$ in Downey's diamond can be c.e.. Indeed, they proved the following lattice embedding:

THEOREM 4 (Ding and Qian [8]). *There are two c.e. degrees* $\boldsymbol{a} < \boldsymbol{b}$ *and a d.c.e. degree* $\boldsymbol{d}$ *such that* $\boldsymbol{d}$ *cups* $\boldsymbol{a}$ *to* $\mathbf{o}'$ *and caps* $\boldsymbol{b}$ *to* $\mathbf{o}$. *Thus,* $\{\mathbf{o}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}, \mathbf{o}'\}$ *is an* N_5 *embedding.*

Obviously, $\{\mathbf{o}, \boldsymbol{a}, \boldsymbol{d}, \mathbf{o}'\}$ in Theorem 4 is a diamond embedding.

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