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REPRESENTABILITY IN SECOND-ORDER PROPOSITIONAL POLY-MODAL LOGIC

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Abstract. A propositional system of modal logic is *second-order* if it contains quantifiers $\forall p$ and $\exists p$, which, in the standard interpretation, are construed as ranging over sets of possible worlds (propositions). Most second-order systems of modal logic are highly intractable; for instance, when augmented with propositional quantifiers, K, B, T, K4 and S4 all become effectively equivalent to full second-order logic. An exception is S5, which, being interpretable in monadic second-order logic, is decidable.

In this paper we generalize this framework by allowing multiple modalities. While this does not affect the undecidability of K, B, T, K4 and S4, poly-modal second-order S5 is dramatically more expressive than its mono-modal counterpart. As an example, we establish the definability of the transitive closure of finitely many modal operators. We also take up the decidability issue, and, using a novel encoding of sets of unordered pairs by partitions of the leaves of certain graphs, we show that the second-order propositional logic of two S5 modalitities is also equivalent to full second-order logic.

§1. Introduction. It is well known that one can extend the language of classical propositional modal logic by adding second-order quantifiers $\forall p$ and $\exists p$. In the standard interpretation these quantifiers range over all *propositions*, i.e., all sets of possible worlds, whereas in the general interpretation the propositional quantifiers range over some collection of propositions that is closed under operations definable in the language. Many systems of propositional modal logic, when augmented with propositional quantifiers, become highly intractable with respect to the standard semantics. For instance, Fine [1970] shows that the systems K, B, T, K4 and S4 become recursively isomorphic to full second order logic upon adjunction of propositional quantifiers.

The system S5, however, is an exception. Independently, Fine [1970] and Kaplan [1970] show that the second-order version of S5 is decidable. The result is established in Fine [1970] by showing how to "eliminate" propositional quantifiers, a proof strategy already employed by Ackermann [1954, pp. 37–47] to show that each sentence of pure monadic second-order logic is equivalent to a first-order sentence in the pure language of identity. (Since the latter theory has the finite model property, monadic second-order logic is decidable.) Kaplan [1970] provides a direct embedding of second-order propositional S5 into monadic second-order logic.

This raises the question of whether the decidability of second-order S5 is due to the symmetric nature of the modality or whether it is due to more peculiar features

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