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PROPERTIES OF FORKING IN ω-FREE PSEUDO-ALGEBRAICALLY CLOSED FIELDS

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Introduction. The study of pseudo-algebraically closed fields (henceforth called PAC) started with the work of J. Ax on finite and pseudo-finite fields [1]. He showed that the infinite models of the theory of finite fields are exactly the perfect PAC fields with absolute Galois group isomorphic to $\hat{\mathbb{Z}}$, and gave elementary invariants for their first order theory, thereby proving the decidability of the theory of finite fields. Ax's results were then extended to a larger class of PAC fields by M. Jarden and U. Kiehne [21], and Jarden [19]. The final word on theories of PAC fields was given by G. Cherlin, L. van den Dries and A. Macintyre [10], see also results by Ju. Ershov [13], [14]. Let K be a PAC field. Then the elementary theory of K is entirely determined by the following data:

- The isomorphism type of the field of absolute numbers of K (the subfield of K of elements algebraic over the prime field).
- The degree of imperfection of K.
- The first-order theory, in a suitable ω-sorted language, of the inverse system of Galois groups $\mathcal{G}al(L/K)$ where L runs over all finite Galois extensions of K.

They also showed that the theory of PAC fields is undecidable, by showing that any graph can be encoded in the absolute Galois group of some PAC field. It turns out that the absolute Galois group controls much of the behaviour of the PAC fields. I will give below some examples illustrating this phenomenon.

The decidability of a class of PAC fields is equivalent to the decidability of the theory of the associated inverse systems of Galois groups. This allowed D. Haran and A. Lubotzky [16] to extend the earlier decidability results on PAC fields with a free Galois group to a larger class of PAC fields, the Frobenius fields: those whose absolute Galois group has the so-called embedding property, see [15] for a definition.

J.-L. Duret showed that PAC fields which are not separably closed are unstable [12]. Most interesting fields (the reals, the p-adics) are unstable, and this led Van den Dries to introduce the concept of algebraically bounded fields in [32]. I will not give the definition here, let me just say that if a field K is algebraically bounded, then one can define a well-behaved dimension on definable sets: if $S \subseteq K^n$ then $\dim(S)$ is the algebraic dimension of the Zariski closure of S in affine n-space. This dimension

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