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DEFINING TRANSCENDENTALS IN FUNCTION FIELDS

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Abstract. Given any field K, there is a function field F/K in one variable containing definable transcendentals over K, i.e., elements in $F \setminus K$ first-order definable in the language of fields with parameters from K. Hence, the model-theoretic and the field-theoretic relative algebraic closure of K in F *do not* coincide. E.g., if K is finite, the model-theoretic algebraic closure of K in the rational function field $K(t)$ is $K(t)$.

For the proof, diophantine $\emptyset$-definability of K in F is established for any function field F/K in one variable, provided K is large, or $K^\times/(K^\times)^n$ is finite for some integer $n > 1$ coprime to *char* K.

§1. Introduction. There are two notions of 'relative algebraic closure' of a field K in a field extension F/K: the **field theoretic relative algebraic closure**

$$\overline{K} \cap F = \{x \in F \mid f(x) = 0 \text{ for some } f \in K[T]\}$$

and the **model theoretic relative algebraic closure**

$$acl_F(K) := \{x \in F \mid x \in A \text{ for some finite } K\text{-definable } A \subseteq F\},$$

where 'K-definable' means definable by a first-order formula in the language of fields with parameters from K. Since polynomial equations over K are instances of such formulas, one always has the inclusion $\overline{K} \cap F \subseteq acl_F(K)$.

The present note provides examples[1] of field extensions F/K where this is a proper inclusion, i.e., where the two notions of relative algebraic closure of K in F *do not* coincide, or, equivalently, where F contains K-definable transcendentals over K. Note that

$$acl_F(K) \not\subseteq \overline{K} \cap F \iff dcl_F(K) \not\subseteq \overline{K} \cap F,$$

where $dcl_F(K) := \{x \in F \mid \{x\} \text{ is } K\text{-definable}\}$ is the **definable closure of K in F**: '$\Leftarrow$' follows from the inclusion $dcl_F(K) \subseteq acl_F(K)$; and, for the converse, if $\{t_1, \dots, t_n\}$ is a finite K-definable subset of F not contained in $\overline{K} \cap F$, then

$$\prod_{i=1}^{n}(T - t_i) \in (dcl_F(K)[T]) \setminus (\overline{K} \cap F)[T].$$

Such examples cannot occur if, e.g., F is algebraically closed or real closed or p-adically closed, or, more generally, if the field-theoretic relative algebraic closure

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