INDESTRUCTIBILITY AND THE LEVEL-BY-LEVEL AGREEMENT BETWEEN STRONG COMPACTNESS AND SUPERCOMPACTNESS

ARTHUR W. APTER[*,†] AND JOEL DAVID HAMKINS[*,†]

Abstract. Can a supercompact cardinal κ be Laver indestructible when there is a level-by-level agreement between strong compactness and supercompactness? In this article, we show that if there is a sufficiently large cardinal above κ, then no, it cannot. Conversely, if one weakens the requirement either by demanding less indestructibility, such as requiring only indestructibility by stratified posets, or less level-by-level agreement, such as requiring it only on measure one sets, then yes, it can.

Two important but apparently unrelated results occupy the large cardinal literature. On the one hand, Laver [Lav78] famously proved that any supercompact cardinal κ can be made indestructible by $<\kappa$-directed closed forcing. On the other hand, Apter and Shelah [AS97] proved that all supercompact cardinals can be preserved to a forcing extension where there is a level-by-level agreement between strong compactness and supercompactness: specifically, except in special cases known to be impossible, any cardinal γ there is η-strongly compact if and only if it is η-supercompact.[1] Can these results be combined? Specifically, we ask:

OPEN QUESTION 1. *Can a supercompact cardinal be indestructible when there is a level-by-level agreement between strong compactness and supercompactness?*

In this article, we provide a partial answer to this question, constraining the possibilities from both above and below. But alas, our results do not settle the matter, so the question remains open. What we can prove, specifically, is that if there is a sufficiently large cardinal above the supercompact cardinal, then the answer to the question is *no*. In particular, there is at most one supercompact cardinal as in the question; more exactly, if a cardinal is indestructibly supercompact in the presence of a level-by-level agreement between strong compactness and supercompactness, then no larger cardinal λ is 2^λ-supercompact. Conversely, if the requirements in the question are weakened in any of several ways, asking either for less indestructibility, replacing it with resurrectibility or with indestructibility by stratified forcing, or for