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ON DIOPHANTINE DEFINABILITY AND DECIDABILITY IN SOME RINGS OF ALGEBRAIC FUNCTIONS OF CHARACTERISTIC 0

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Abstract. Let K be a function field of one variable over a constant field C of finite transcendence degree over $\mathbb{C}$. Let M/K be a finite extension and let W be a set of primes of K such that all but finitely many primes of W do not split in the extension M/K. Then there exists a set W' of K-primes such that Hilbert's Tenth Problem is not decidable over $O_{K,W'} = \{x \in K | \text{ord}_{\mathfrak{p}} x \geq 0, \forall \mathfrak{p} \notin W'\}$, and the set $(W' \setminus W) \cup (W \setminus W')$ is finite.

Let K be a function field of one variable over a constant field C finitely generated over $\mathbb{Q}$. Let M/K be a finite extension and let W be a set of primes of K such that all but finitely many primes of W do not split in the extension M/K and the degree of all the primes in W is bounded by $b \in \mathbb{N}$. Then there exists a set W' of K-primes such that $\mathbb{Z}$ has a Diophantine definition over $O_{K,W'}$, and the set $(W' \setminus W) \cup (W \setminus W')$ is finite.

§1. Introduction. The interest in the questions of Diophantine definability and decidability goes back to a question which was posed by Hilbert: given an arbitrary polynomial equation in several variables over $\mathbb{Z}$, is there a uniform algorithm to determine whether such an equation has solutions in $\mathbb{Z}$? This question, otherwise known as Hilbert's 10th problem, has been answered negatively in the work of M. Davis, H. Putnam, J. Robinson and Yu. Matijasevich. (See [3] and [4].) Since the time when this result was obtained, similar questions have been raised for other fields and rings. Arguably the two most interesting and difficult problems in the area are the questions of Diophantine decidability of $\mathbb{Q}$ and the rings of algebraic integers of arbitrary number fields. One way to resolve the question of Diophantine decidability negatively over a ring of characteristic 0 is to construct a Diophantine definition of $\mathbb{Z}$ over such a ring. This notion is defined below.

DEFINITION 1.1. Let R be a ring and let $A \subset R$. Then we say that A has a Diophantine definition over R if there exists a polynomial $f(t, x_1, \ldots, x_n) \in R[t, x_1, \ldots, x_n]$ such that for any $t \in R$,

$$\exists x_1, \ldots, x_n \in R, f(t, x_1, ..., x_n) = 0 \Longleftrightarrow t \in A.$$

If the quotient field of R is not algebraically closed, it can be shown that we can allow Diophantine definitions to consist of several polynomials without changing

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