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DEFINABLE INCOMPLETENESS AND FRIEDBERG SPLITTINGS

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Abstract. We define a property $R(A_0, A_1)$ in the partial order $\mathscr{E}$ of computably enumerable sets under inclusion, and prove that R implies that A_0 is noncomputable and incomplete. Moreover, the property is nonvacuous, and the A_0 and A_1 which we build satisfying R form a Friedberg splitting of their union A, with A_1 prompt and A promptly simple. We conclude that A_0 and A_1 lie in distinct orbits under automorphisms of $\mathscr{E}$, yielding a strong answer to a question previously explored by Downey, Stob, and Soare about whether halves of Friedberg splittings must lie in the same orbit.

§1. Introduction. The computably enumerable sets form an upper semi-lattice under Turing reducibility. Under set inclusion, they form a lattice $\mathscr{E}$, as first noted by Myhill in [14], and the properties of a c. e. set as an element of $\mathscr{E}$ often help determine its properties under Turing reducibility. Even before Myhill, Post had suggested that there should be a nonvacuous property of c. e. sets, definable without reference to the Turing degrees, which would imply that the Turing degree of such a set must lie strictly between the computable degree $\mathbf{0}$ and the complete c. e. degree $\mathbf{0}'$.

Post's own attempts to find such a property failed. The properties he defined turned out to be extremely useful in computability theory, but each of them—simplicity, hypersimplicity, and hyperhypersimplicity—actually does hold of some complete set. The existence of a Turing degree between $\mathbf{0}$ and $\mathbf{0}'$ was first proven by completely different means, namely the finite injury constructions of Friedberg and Muchnik ([6], [13]).

The term "Post's Program" eventually came to denote the search for an $\mathscr{E}$-definable property implying incompleteness. Of the properties proposed by Post, all except hypersimplicity turned out to be definable in $\mathscr{E}$, and other $\mathscr{E}$-definable properties, such as maximality, were developed and studied in their own right. Nevertheless, Post's Program remained unfinished until 1991, when Harrington and Soare ([7]) found a property $Q(A)$ definable in $\mathscr{E}$ such that every A satisfying Q must be both noncomputable and Turing-incomplete. We give their definition of $Q(A)$:

$$\begin{aligned} Q(A) : (\exists C)_{A \subset_{\mathrm{m}} C}\ (\forall B \subseteq C)\ (\exists D \subseteq C)\ (\forall S)_{S \sqsubset C} \\ \big(B \cap (S - A) = D \cap (S - A) \\ \implies (\exists T)\ [\overline{C} \subset T\ \&\ A \cap (S \cap T) = B \cap (S \cap T)]\big). \end{aligned}$$

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