THE JOURNAL OF SYMBOLIC LOGIC
Volume 67, Number 2, June 2002

CODING WITH LADDERS A WELL ORDERING OF THE REALS

URI ABRAHAM AND SAHARON SHELAH

Abstract. Any model of ZFC + GCH has a generic extension (made with a poset of size $\aleph_2$) in which the following hold: $\mathrm{MA} + 2^{\aleph_0} = \aleph_2 +$ *there exists a* Δ^2_1-*well ordering of the reals.* The proof consists in iterating posets designed to change at will the guessing properties of ladder systems on ω_1. Therefore, the study of such ladders is a main concern of this article.

§1. Preface. The character of possible well-orderings of the reals is a main theme in set theory, and the work on long projective well-orderings by L. Harrington [4] can be cited as an example. There, the relative consistency of $\mathrm{ZFC}+\mathrm{MA}+2^{\aleph_0} > \aleph_1$ with the existence of a Δ^1_3 well-ordering of the reals is shown. A different type of question is to ask about the impact of large cardinals on definable well-orderings. Work of Shelah and Woodin [7], and Woodin [9] is relevant to this type of question. Assuming in V a cardinal which is both measurable and Woodin, Woodin [9] proved that if CH holds, then there is no Σ^2_1 well-ordering of the reals. This result raises two questions:

1. If large cardinals and CH are assumed in V, can the Σ^2_1 result be strengthened to Σ^2_2? That is, is there a proof that large cardinals and CH imply there are no Σ^2_2 well-orderings of the reals?
2. What happens if CH is not assumed?

Regarding the first question, Abraham and Shelah [2] describes a poset of size $\aleph_2$ (assuming GCH) which generically adds no reals and provides a Δ^2_2 well-ordering of the reals. Thus, if one starts with any universe with a large cardinal κ, one can extend this universe with a small size forcing and obtain a Δ^2_2 well-ordering of the reals. Since small forcings will not alter the assumed largeness of a cardinal in V, the answer to question 1 is negative.

Regarding the second question, Woodin (unpublished) uses an inaccessible cardinal κ to obtain a generic extension in which

1. MA for σ-centered posets $+ 2^{\aleph_0} = \kappa$, and
2. there is a Σ^2_1 well-ordering of the reals.

Solovay [8] shows that the inaccessible cardinal is dispensable: any model of ZFC has a small size forcing extension in which the following holds:

Received October 18, 1998; revised March 21, 2001.
This research of the second author was supported by The Israel Science Foundation founded by the Israel Academy of Sciences and Humanities. Publication #485.

0022-4812/02/6702-0007/$2.90