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CONGRUENCE RELATIONS ON LATTICES OF RECURSIVELY ENUMERABLE SETS

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§1. Introduction. Let $\{W_e\}_{e\in\omega}$ be a standard enumeration of the recursively enumerable (r. e.) subsets of $\omega = \{0, 1, 2, \dots\}$. The lattice of recursively enumerable sets, $\mathscr{E}$, is the structure $(\{W_e\}_{e\in\omega}, \cup, \cap)$. $\mathscr{R}$ is the sublattice of $\mathscr{E}$ consisting of the recursive sets.

Suppose $\mathscr{U}$ is a lattice of subsets of ω. $\equiv$ is said to be a congruence relation on $\mathscr{U}$ if $\equiv$ is an equivalence relation on $\mathscr{U}$ and if for all U, $U' \in \mathscr{U}$ and V, $V' \in \mathscr{U}$, if $U \equiv U'$ and $V \equiv V'$, then $U \cup U' \equiv V \cup V'$ and $U \cap U' \equiv V \cap V'$. $[U] = \{ V \in \mathscr{U} \mid V \equiv U \}$ is the equivalence class of U. If $\equiv$ is a congruence relation on $\mathscr{U}$, the elements of the quotient lattice $\mathscr{E}/{\equiv}$ are the equivalence classes of $\equiv$. $[U] \cup [V]$ is defined as $[U \cup V]$, and $[U] \cap [V]$ is defined as $[U \cap V]$.

The quotient lattices of $\mathscr{E}$ (or of some sublattice $\mathscr{U}$) correspond naturally with the congruence relations which give rise to them, and in turn the congruence relations of sublattices of $\mathscr{E}$ can be characterized in part by their computational complexity. The aim of the present paper is to characterize congruence relations in some of the most important complexity classes.

A few simple but important congruence relations can be defined on any lattice $\mathscr{U}$ of subsets of ω. The congruence relation $=^*$ is defined by putting $U =^* V$ if and only if $U \,\Delta\, V$ is finite, where $U \,\Delta\, V = (U \cap \overline{V}) \cup (\overline{U} \cap V)$ is the symmetric difference of U and V. If X is any subset of ω, we define the congruence relation $=_X$ by putting $U =_X V$ if and only if $U \cap X = V \cap X$. Similarly, the congruence relation $=^*_X$ is defined by putting $U =^*_X V$ if and only if $U \cap X =^* V \cap X$.

An important theme in the study of the recursively enumerable sets has been to show that increasingly large classes of quotient lattices $\mathscr{E}/{\equiv}$ share many of the algebraic properties of $\mathscr{E}$. One particularly important line of research started with the result of Friedberg [3] that there exists a maximal element in the quotient lattice $\mathscr{E}/{=^*}$. The lattice $\mathscr{E}/{=^*}$ is usually written $\mathscr{E}^*$. Robinson [11] extended Friedberg's result to prove that for any coinfinite low r. e. set A, there exists a maximal element in the quotient lattice $\mathscr{E}/{=^*_{\overline{A}}}$. ($A$ is said to be low if $A' \equiv_T \emptyset'$, where A' is the Turing jump of A, and where $\equiv_T$ is Turing equivalence; see Soare [13] for more details.) The lattice $\mathscr{E}/{=^*_X}$ is often denoted $\mathscr{E}^*(X)$. Bennison and Soare [1] extended Robinson's

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