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HOW TO EXTEND THE SEMANTIC TABLEAUX AND CUT-FREE VERSIONS OF THE SECOND INCOMPLETENESS THEOREM ALMOST TO ROBINSON'S ARITHMETIC Q

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Abstract. Let us recall that Raphael Robinson's Arithmetic Q is an axiom system that differs from Peano Arithmetic essentially by containing no Induction axioms [13], [18]. We will generalize the semantic-tableaux version of the Second Incompleteness Theorem *almost to the level* of System Q. We will prove that there exists a single rather long Π_1 sentence, valid in the standard model of the Natural Numbers and denoted as V, such that if α is *any* finite consistent extension of $Q + V$ then α will be unable to prove its Semantic Tableaux consistency. The same result will also apply to axiom systems α with infinite cardinality when these infinite-sized axiom systems satisfy a minor additional constraint, called the *Conventional Encoding Property*.

Our formalism will also imply that the semantic-tableaux version of the Second Incompleteness Theorem generalizes for the axiom system $I\Sigma_0$, as well as for all its natural extensions. (This answers an open question raised twenty years ago by Paris and Wilkie [15].)

§1. Introduction. As originally formulated by Gödel, the Second Incompleteness Theorem discussed the inability of any extension of Peano Arithmetic to verify its own consistency when its proofs were constructed using a deductive calculi similar to a Hilbert (or Frege) type formalism. When Smullyan introduced his Semantic Tableaux version [20] of Gentzen's cut-free Sequent Calculus, it was realized that all extensions of Peano Arithmetic would also be unable to verify their own consistency if the Semantic Tableaux trees replaced the Frege-Hilbert methodologies as the underlying deductive calculi for generating the formal proofs.

But what about axiom systems weaker than Peano Arithmetic? *Would they also be inherently unable to verify their own consistency under the Hilbert and Semantic Tableaux styles of deductive calculi?*

Some initial research into this question was done by Bezboruah and Shepherdson [5], but it required a surprising 54 years after the publication of Gödel's Incompleteness Theorem for mathematicians to begin to formulate a full answer to it. Let us recall that Robinson's Arithmetic Q differs from Peano Arithmetic by containing no Induction axioms [13], [18]. In 1985, Pudlák proved [16] that no conceivable extension α of Q could formally verify its own consistency when a Hilbert-style form of deductive calculi was used to generate the formal proof structures. Moreover, Robert Solovay (private communications, [21]) combined several formalisms

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