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PFAFFIAN DIFFERENTIAL EQUATIONS OVER EXPONENTIAL O-MINIMAL STRUCTURES

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In this paper, we continue investigations into the asymptotic behavior of solutions of differential equations over o-minimal structures.

Let $\mathfrak{R}$ be an expansion of the real field $(\mathbb{R}, +, \cdot)$.

A differentiable map $F = (F_1, \ldots, F_l)\colon (a, b) \to \mathbb{R}^l$ is **$\mathfrak{R}$-Pfaffian** if there exists $G\colon \mathbb{R}^{1+l} \to \mathbb{R}^l$ definable in $\mathfrak{R}$ such that $F'(t) = G(t, F(t))$ for all $t \in (a, b)$ and each component function $G_i\colon \mathbb{R}^{1+l} \to \mathbb{R}$ is independent of the last $l - i$ variables $(i = 1, \ldots, l)$. If $\mathfrak{R}$ is o-minimal and $F\colon (a, b) \to \mathbb{R}^l$ is $\mathfrak{R}$-Pfaffian, then $(\mathfrak{R}, F)$ is o-minimal (Proposition 7). We say that $F\colon \mathbb{R} \to \mathbb{R}^l$ is ultimately $\mathfrak{R}$-Pfaffian if there exists $r \in \mathbb{R}$ such that the restriction $F{\restriction}(r, \infty)$ is $\mathfrak{R}$-Pfaffian. (In general, **ultimately** abbreviates "for all sufficiently large positive arguments".)

The structure $\mathfrak{R}$ is **closed under asymptotic integration** if for each ultimately nonzero unary (that is, $\mathbb{R} \to \mathbb{R}$) function f definable in $\mathfrak{R}$ there is an ultimately differentiable unary function g definable in $\mathfrak{R}$ such that $\lim_{t\to+\infty}[g'(t)/f(t)] = 1$. If $\mathfrak{R}$ is closed under asymptotic integration, then $\mathfrak{R}$ is o-minimal and defines $e^x\colon \mathbb{R} \to \mathbb{R}$ (Proposition 2).

Note that the above definitions make sense for expansions of arbitrary ordered fields.

THEOREM 1. *If $\mathfrak{R}$ is o-minimal, then the following are equivalent*:

1. *For every ultimately $\mathfrak{R}$-Pfaffian function $F\colon \mathbb{R} \to \mathbb{R}$ there exists $u\colon \mathbb{R} \to \mathbb{R}$ definable in $\mathfrak{R}$ such that ultimately $F(t) \le u(t)$.*
2. *$\mathfrak{R}$ is closed under asymptotic integration.*
3. *Every structure elementarily equivalent to $\mathfrak{R}$ is closed under asymptotic integration.*
4. *For every $m \in \mathbb{N}$ and $f\colon \mathbb{R}^{m+1} \to \mathbb{R}$ definable in $\mathfrak{R}$ there exists $u\colon \mathbb{R} \to \mathbb{R}$ definable in $\mathfrak{R}$ such that* $\lim_{t\to+\infty} u(t) = +\infty$ *and, for all $a \in \mathbb{R}^m$,*
$$\lim_{t\to+\infty} f(a,t)u'(t) = 0 \quad \text{or} \quad \lim_{t\to+\infty} |f(a,t)(1/u)'(t)| = +\infty.$$
5. *For every $l \in \mathbb{N}$, ultimately $\mathfrak{R}$-Pfaffian $F\colon \mathbb{R} \to \mathbb{R}^l$ and $h\colon \mathbb{R}^{1+l} \to \mathbb{R}$ definable in $\mathfrak{R}$ there exists $u\colon \mathbb{R} \to \mathbb{R}$ definable in $\mathfrak{R}$ such that ultimately $h(t, F(t)) \le u(t)$.*

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