The Journal of Symbolic Logic
Volume 67, Number 1, March 2002

ON ESSENTIALLY LOW, CANONICALLY WELL-GENERATED BOOLEAN ALGEBRAS

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Abstract. Let B be a superatomic Boolean algebra (BA). The rank of B (rk(B)), is defined to be the Cantor Bendixon rank of the Stone space of B. If $a \in B - \{0\}$, then the rank of a in B (rk(a)), is defined to be the rank of the Boolean algebra $B \restriction a \stackrel{\text{def}}{=} \{b \in B : b \leq a\}$. The rank of 0^B is defined to be -1. An element $a \in B - \{0\}$ is a generalized atom ($a \in \widehat{\mathrm{At}}(B)$), if the last nonzero cardinal in the cardinal sequence of $B \restriction a$ is 1. Let $a, b \in \widehat{\mathrm{At}}(B)$. We denote $a \sim b$, if $\mathrm{rk}(a) = \mathrm{rk}(b) = \mathrm{rk}(a \cdot b)$. A subset $H \subseteq \widehat{\mathrm{At}}(B)$ is a complete set of representatives (CSR) for B, if for every $a \in \widehat{\mathrm{At}}(B)$ there is a unique $h \in H$ such that $h \sim a$. Any CSR for B generates B. We say that B is canonically well-generated (CWG), if it has a CSR H such that the sublattice of B generated by H is well-founded. We say that B is well-generated, if it has a well-founded sublattice L such that L generates B.

Theorem 1. Let B be a Boolean algebra with cardinal sequence $\langle \aleph_0 : i < \alpha \rangle^\frown \langle \lambda, 1 \rangle$, $\alpha < \aleph_1$. If B is CWG, then every subalgebra of B is CWG.

A superatomic Boolean algebra B is essentially low (ESL), if it has a countable ideal I such that $\mathrm{rk}(B/I) \leq 1$.

Theorem 1 follows from Theorem 2.9, which is the main result of this work. For an ESL BA B we define a set F^B of partial functions from a certain countably infinite set to ω (Definition 2.8). Theorem 2.9 says that if B is an ESL Boolean algebra, then the following are equivalent.
(1) Every subalgebra of B is CWG; and
(2) F^B is bounded.

Theorem 2. If an ESL Boolean algebra is not CWG, then it has a subalgebra which is not well-generated.

§1. Introduction.

We consider a certain class of superatomic Boolean algebras, which we call "essentially low" Boolean algebras. We say that a superatomic Boolean algebra B is "essentially low" (B is ESL), if either $|B| \leq \aleph_0$ or B has a countable ideal I such that B/I is isomorphic to a Boolean algebra of the form $B_0(\mu_1) \times \cdots \times B_0(\mu_n)$, where each μ_i is an uncountable cardinal, and $B_0(\mu_i)$ denotes the algebra of all finite and cofinite subsets of μ_i. ESL algebras arise in the study of Boolean algebras with cardinal sequence $\langle \aleph_0 : i < \alpha \rangle^\frown \langle \lambda, 1 \rangle$, α countable. Every ESL algebra is embeddable in an algebra of the form $B_1 \times B_0(\mu_1) \times \cdots \times B_0(\mu_n)$, where B_1 is an ESL subalgebra of $\wp(\omega)$. So essentially this work is a study of a class of subalgebras of $\wp(\omega)$.

Received September 14, 1999; revised May 2, 2001
1991 *Mathematics Subject Classification.* Primary: 03E04, 06E05.
Key words and phrases. superatomic Boolean algebras, bounding number of $\wp(\omega)$.

0022-4812/02/6701-0025/$3.80