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THE RELATIVE CONSISTENCY OF $\mathfrak{g} < \mathrm{cf}(\mathrm{Sym}(\omega))$

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Abstract. We prove the consistency result from the title. By forcing we construct a model of $\mathfrak{g} = \aleph_1$, $\mathfrak{b} = \mathrm{cf}(\mathrm{Sym}(\omega)) = \aleph_2$.

§1. Introduction. We recall the definitions of the three cardinal characteristics in the title and the abstract. We write $A \subseteq^* B$ if $A \setminus B$ is finite. We write $f \leq^* g$ if $f, g \in {}^\omega\omega$ and $\{n : f(n) > g(n)\}$ is finite.

DEFINITION 1.1. (1) *A subset $\mathscr{G}$ of $[\omega]^\omega$ is called groupwise dense if*

– for all $B \in \mathscr{G}$, $A \subseteq^* B$ we have that $A \in \mathscr{G}$ and

– for every partition $\{[\pi_i, \pi_{i+1}) : i \in \omega\}$ of ω into finite intervals there is an infinite set A such that $\bigcup\{[\pi_i, \pi_{i+1}) : i \in A\} \in \mathscr{G}$.

The groupwise density number, $\mathfrak{g}$, is the smallest number of groupwise dense families with empty intersection.

(2) $\mathrm{Sym}(\omega)$ *is the group of all permutations of ω. If* $\mathrm{Sym}(\omega) = \bigcup_{i<\kappa} K_i$ *and $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, $\langle K_i : i < \kappa\rangle$ is increasing and continuous, K_i is a proper subgroup of* $\mathrm{Sym}(\omega)$, *we call $\langle K_i : i < \kappa\rangle$ a cofinality witness. We call the minimal such κ the cofinality of the symmetric group, short* $\mathrm{cf}(\mathrm{Sym}(\omega))$.

(3) *The bounding number $\mathfrak{b}$ is*

$$\mathfrak{b} = \min\{|\mathscr{F}| : \mathscr{F} \subseteq {}^\omega\omega \wedge (\forall g \in {}^\omega\omega)(\exists f \in \mathscr{F}) f \not\leq^* g\}.$$

Simon Thomas asked whether $\mathfrak{g} \neq \mathrm{cf}(\mathrm{Sym}(\omega))$ is consistent [9, Question 3.1]. In this work we prove:

THEOREM 1.2. $\mathfrak{g} < \mathrm{cf}(\mathrm{Sym}(\omega))$ *is consistent relative to* ZFC.

§2. Forcings destroying many cofinality witnesses. In this section we introduce two families of forcings that will be used in certain steps of our planned iteration of length $\aleph_2$. The plot is: If $\mathfrak{b}$ is large, there is some way to destroy all shorter cofinality witnesses because by Claims 2.6 and 2.5 none of the subgroups in a cofinality witness contains all permutations respecting a given equivalence relation.

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