THE JOURNAL OF SYMBOLIC LOGIC
Volume 67, Number 1, March 2002

PROVING CONSISTENCY OF EQUATIONAL THEORIES IN BOUNDED ARITHMETIC

ARNOLD BECKMANN†

Abstract. We consider equational theories for functions defined via recursion involving equations between closed terms with natural rules based on recursive definitions of the function symbols. We show that consistency of such equational theories can be proved in the weak fragment of arithmetic S^1_2. In particular this solves an open problem formulated by TAKEUTI (c.f. [5, p.5 problem 9.]).

§1. **Introduction.** Since the introduction of bounded arithmetic it has been and still is a major open problem if bounded arithmetic is finitely axiomatizable, or, equivalently, if the hierarchy of bounded arithmetic theories S^i_2 (c.f. [2]) is proper. One of the first ideas to attack this problem which comes into one's mind is to use consistency statements as separating sentences. However, up to now only negative results have been achieved in this direction. The usual notion of consistency is too strong as $S_2 \nvdash Con_{S_2^{-1}}$, c.f. [10], where S_2^{-1} is the induction-free fragment of bounded arithmetic S_2. Also the weaker consistency statements $BDCon$ which refer to proofs that use only bounded formulas still is too strong: S. BUSS [2] proved that $S_2^{i+1} \vdash BDCon_{S^i_2}$ holds for at most one i, and P. PUDLÁK showed in [8] that $S_2 \nvdash BDCon_{S^1_2}$, hence only $S_2 \vdash BDCon_{S^0_2}$ remains to be possible. The reason why usual approaches for proving consistency do not work in weak arithmetic is that it is impossible to feasibly evaluate closed terms from the language of bounded arithmetic – their values grow exponentially in their GÖDEL-numbers in general. This leads to the plausible conjecture raised by G. TAKEUTI, c.f. [5, p.5 problem 9.]: "Let $S_2^{-\infty}$ be the equational theory involving equations $s = t$, where s, t are closed terms in the language of S_2, with natural rules based on recursive definitions of the function symbols. Show that $S_2 \nvdash Con(S_2^{-\infty})$. [...]"

If this conjecture would be true then it would be likely that consistency statements cannot be used to negatively answer the finitely axiomatizability problem of bounded arithmetic. In this paper we will disprove this conjecture, thus there is hope that consistency statements can lead to a negative answer.

More generally we will consider equational theories for functions defined via recursion involving equations between closed terms with natural rules based on recursive definitions of the function symbols. The recursion can be defined very

Received May 14, 1999; accepted February 10, 2001.
†Supported by the Deutschen Akademie der Naturforscher Leopoldina grant #BMBF-LPD 9801-7 with funds from the Bundesministeriums für Bildung, Wissenschaft, Forschung und Technologie.

0022-4812/02/6701-0019/$2.80