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WELLORDERING PROOFS FOR METAPREDICATIVE MAHLO

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Abstract. In this article we provide wellordering proofs for metapredicative systems of explicit mathematics and admissible set theory featuring suitable axioms about the Mahloness of the underlying universe of discourse. In particular, it is shown that in the corresponding theories EMA of explicit mathematics and KPm^0 of admissible set theory, transfinite induction along initial segments of the ordinal $\varphi\omega 00$, for φ being a ternary Veblen function, is derivable. This reveals that the upper bounds given for these two systems in the paper Jäger and Strahm [11] are indeed sharp.

§1. Introduction. This paper is a companion to the article by Jäger and Strahm [11], where systems of explicit mathematics and admissible set theory for metapredicative Mahlo are introduced. Whereas the main concern of [11] was to establish proof-theoretic upper bounds for these systems, in this article we provide the corresponding wellordering proofs, thus showing that the upper bounds derived in [11] are sharp.

The central systems of this article are the theories EMA and KPm^0 for metapredicative Mahlo in explicit mathematics and admissible set theory, respectively. EMA is based on Feferman's explicit mathematics with elementary comprehension and join (c.f., Feferman [2, 3]). Crucial for its formulation are so-called *universes*: these are types of representations or names which are closed under elementary comprehension and join. The principal axiom of EMA claims that for each operation from names of types to names of types there exists a uniformly given universe that is closed under this operation. We note that EMA does not include inductive generation and that induction on the natural number is restricted to types. For more information concerning EMA plus inductive generation see Jäger and Studer [12].

The theory KPm^0, on the other hand, is Rathjen's theory KPM (c.f., Rathjen [16, 17]) with induction on the natural numbers restricted to sets and $\in$ induction omitted completely. The Mahlo axiom schema in KPm^0 features Π_2 reflection on admissible sets. It happens that the absence of $\in$ induction causes a dramatic collapse in proof-theoretic strength: whereas KPM is a highly impredicative theory exceeding $(\Delta^1_2\text{-CA})+(\mathsf{BI})$ in proof strength by far, the strength of KPm^0 is between the Feferman-Schütte ordinal Γ_0 and the Bachmann-Howard ordinal.

The theories EMA and KPm^0 and their proof-theoretic analyses typically belong to the new area of so-called *metapredicative proof theory*. Metapredicativity is concerned with the study and analysis of formal systems whose proof-theoretic

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