RELATION ALGEBRA REDUCTS OF CYLINDRIC ALGEBRAS
AND AN APPLICATION TO PROOF THEORY

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Abstract. We confirm a conjecture, about neat embeddings of cylindric algebras, made in 1969 by
J. D. Monk, and a later conjecture by Maddux about relation algebras obtained from cylindric algebras.
These results in algebraic logic have the following consequence for predicate logic: for every finite cardinal
α ≥ 3 there is a logically valid sentence X, in a first-order language \( \mathcal{L} \) with equality and exactly one
nonlogical binary relation symbol E, such that X contains only 3 variables (each of which may occur
arbitrarily many times). X has a proof containing exactly \( \alpha + 1 \) variables, but X has no proof containing
only \( \alpha \) variables. This solves a problem posed by Tarski and Givant in 1987.

§1. Introduction. The completeness theorem of first-order logic says that every
valid formula has a proof. However, results of Henkin and Monk showed that
the proof of a formula may need more variables than are used in the formula
itself. Establishing exactly how many variables are needed to prove a given valid
formula can be rather delicate. To establish provability or non-provability with \( \alpha \)
variables, the methods of algebraic logic — cylindric algebras and relation algebras
— are useful. \( \alpha \)-dimensional cylindric algebras can be regarded, approximately, as
algebras of \( \alpha \)-ary relations and relation algebras are an algebraic approximation
to algebras of binary relations. From an \( \alpha \)-dimensional cylindric algebra \( \mathcal{C} \) it is
possible to obtain the relation algebra reduct \( \mathcal{R}(\alpha) \mathcal{C} \), and if \( \alpha \geq 4 \) this will be a
relation algebra. The central part of this paper is the construction of some relation
algebras \( \mathcal{N}_\alpha^\beta \), for \( 4 \leq \alpha \leq \beta < \omega \), and the proof, for sufficiently large \( \beta \),
that \( \mathcal{N}_\alpha^\beta \) is a subalgebra of \( \mathcal{R}(\alpha) \mathcal{C} \) for some \( \alpha \)-dimensional cylindric algebra \( \mathcal{C} \), but not a
subalgebra of \( \mathcal{R}(\alpha + 1) \mathcal{C} \) for any \( (\alpha + 1) \)-dimensional cylindric algebra \( \mathcal{C} \). In symbols,
\( \mathcal{N}_\alpha^\beta \in S \mathcal{R}(\alpha) \mathcal{C} \setminus S \mathcal{R}(\alpha + 1) \mathcal{C} \). This confirms a conjecture of Maddux, and is used to
confirm a related conjecture of Monk about neat reducts of cylindric algebras. We
apply this result to logic by showing, for each \( \alpha \geq 3 \), that there are valid formulas
that can be proved with \( \alpha + 1 \) variables but not with only \( \alpha \) variables in a proof
system taken from [31].

Here in the introduction we discuss these classes of algebras, some of the history
of this investigation, and the proof-theoretic consequences. In the second section