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# RELATION ALGEBRA REDUCTS OF CYLINDRIC ALGEBRAS AND AN APPLICATION TO PROOF THEORY

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**Abstract.** We confirm a conjecture, about neat embeddings of cylindric algebras, made in 1969 by J. D. Monk, and a later conjecture by Maddux about relation algebras obtained from cylindric algebras. These results in algebraic logic have the following consequence for predicate logic: for every finite cardinal $\alpha \geq 3$ there is a logically valid sentence $X$, in a first-order language $\mathscr{L}$ with equality and exactly one nonlogical binary relation symbol $\boldsymbol{E}$, such that $X$ contains only 3 variables (each of which may occur arbitrarily many times), $X$ has a proof containing exactly $\alpha + 1$ variables, but $X$ has no proof containing only $\alpha$ variables. This solves a problem posed by Tarski and Givant in 1987.

§1. **Introduction.** The completeness theorem of first-order logic says that every valid formula has a proof. However, results of Henkin and Monk showed that the proof of a formula may need more variables than are used in the formula itself. Establishing exactly how many variables are needed to prove a given valid formula can be rather delicate. To establish provability or non-provability with $\alpha$ variables, the methods of algebraic logic — cylindric algebras and relation algebras — are useful. $\alpha$-dimensional cylindric algebras can be regarded, approximately, as algebras of $\alpha$-ary relations and relation algebras are an algebraic approximation to algebras of binary relations. From an $\alpha$-dimensional cylindric algebra $\mathfrak{C}$ it is possible to obtain the *relation algebra reduct* $\mathfrak{Ra}\,\mathfrak{C}$, and if $\alpha \geq 4$ this will be a relation algebra. The central part of this paper is the construction of some relation algebras $\mathfrak{N}^{\beta}_{\alpha}$, for $4 \leq \alpha \leq \beta < \omega$, and the proof, for sufficiently large $\beta$, that $\mathfrak{N}^{\beta}_{\alpha}$ is a subalgebra of $\mathfrak{Ra}\,\mathfrak{C}$ for some $\alpha$-dimensional cylindric algebra $\mathfrak{C}$, but not a subalgebra of $\mathfrak{Ra}\,\mathfrak{C}'$ for any $(\alpha+1)$-dimensional cylindric algebra $\mathfrak{C}'$. In symbols, $\mathfrak{N}^{\beta}_{\alpha} \in \boldsymbol{S}\,\mathsf{Ra}\,\mathsf{CA}_{\alpha} \setminus \boldsymbol{S}\,\mathsf{Ra}\,\mathsf{CA}_{\alpha+1}$. This confirms a conjecture of Maddux, and is used to confirm a related conjecture of Monk about neat reducts of cylindric algebras. We apply this result to logic by showing, for each $\alpha \geq 3$, that there are valid formulas that can be proved with $\alpha + 1$ variables but not with only $\alpha$ variables in a proof system taken from [31].

Here in the introduction we discuss these classes of algebras, some of the history of this investigation, and the proof-theoretic consequences. In the second section

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