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EMBEDDING FINITE LATTICES INTO THE Σ^0_2 ENUMERATION DEGREES

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Abstract. We show that every finite lattice is embeddable into the Σ^0_2 enumeration degrees via a lattice-theoretic embedding which preserves 0 and 1.

§1. Introduction. Informally, a set A is enumeration reducible to a set B if there is some effective procedure for enumerating A, given any enumeration of B. This informal notion of reducibility can be formalized using the notion of enumeration operator. Let $\{W_i\}_{i\in\omega}$ be the standard listing of the computably enumerable (c.e.) sets. With every c.e. set W_i, one can associate a mapping $\Phi_i : P(\omega) \to P(\omega)$ (where $P(\omega)$ is the power set of the set of natural numbers ω) by letting, for every B,

$$\Phi_i^B = \{x : (\exists u)[\langle x, u\rangle \in W_i \;\&\; D_u \subseteq B]\}$$

(where $\langle \cdot, \cdot\rangle$ is the usual pairing function, providing a computable one-one bijection of ω^2 onto ω; and D_u is the finite set with canonical index u, i.e., D_u denotes the finite set D for which $u = \sum_{x\in D} 2^x$; see e.g., [17]. In the following, finite sets will be often identified with their canonical indices). A mapping $\Phi : P(\omega) \to P(\omega)$ is called an *enumeration operator* (or simply an *e-operator*) if $\Phi = \Phi_i$ for some i.

Given sets of numbers A and B, we say that A is *enumeration reducible* (or simply *e-reducible*) to B if $A = \Phi^B$ for some e-operator Φ. This reducibility is easily seen to be a partial preordering relation, which will be denoted by the symbol $\leq_e$.

The degree structure induced by $\leq_e$ is the structure of the *enumeration degrees* (simply *e-degrees*), denoted by $\mathfrak{D}_e$. The e-degree of a set X will be denoted by $\deg_e(X)$. $\mathfrak{D}_e$ is in fact an upper semilattice with least element $\boldsymbol{0}_e$, with $\boldsymbol{0}_e = \deg_e(W)$ where W is any c.e. set. It is known (Gutteridge, see also [6]) that $\mathfrak{D}_e$ does not have minimal elements (although the structure is not dense, see [8]; Calhoun and Slaman in [5], have shown that there exist Π^0_2 e-degrees $\boldsymbol{a} < \boldsymbol{b}$ such that $\boldsymbol{b}$ is a minimal cover of $\boldsymbol{a}$). An important substructure of $\mathfrak{D}_e$ is given by the Σ^0_2 e-degrees, i.e., the e-degrees of the Σ^0_2 sets. Let $\mathfrak{S}$ denote the structure of the e-degrees of the Σ^0_2 sets. Cooper [7] shows that $\mathfrak{S} = \mathfrak{D}_e(\leq_e \boldsymbol{0}'_e)$ where $\boldsymbol{0}'_e = \deg_e(\overline{K})$, $\overline{K}$ being the complement of the halting set (for a definition of the jump operation on the e-degrees, see [7] and [13]). Cooper [7] shows that $\mathfrak{S}$ is dense.

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