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THE COST OF A CYCLE IS A SQUARE

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Abstract. The logical flow graphs of sequent calculus proofs might contain oriented cycles. For the predicate calculus the elimination of cycles might be non-elementary and this was shown in [Car96]. For the propositional calculus, we prove that if a proof of k lines contains n cycles then there exists an acyclic proof with $\mathcal{O}(k^{n+1})$ lines. In particular, there is a polynomial time algorithm which eliminates cycles from a proof. These results are motivated by the search for general methods on proving lower bounds on proof size and by the design of more efficient heuristic algorithms for proof search.

§1. Introduction. A formal proof presents an underlying graph structure which is induced by the flow of its formula occurrences. By applying formal rules one after the other, one implicitly defines logical relations between the formula occurrences in a deduction, and by tracing the graph of these relations one usually obtains rather complicated graphs.

Different formal systems correspond to different properties of such graphs. The logical graphs of resolution proofs and cut-free proofs for instance, are essentially *trees* (or better say, collections of trees), while proofs with cuts might present instead a quite intricate 'topology'. In particular, they might contain cycles and at times nested loops. A study of the combinatorial properties of graphs of proofs with cuts is done in [Car97b].

In [Bus91], Buss introduces the idea of tracing formula occurrences in a proof and he calls *logical flow graph* the underlying graph structure of a proof. In [Bus93], Buss observes that logical flow graphs might contain oriented cycles. In [Car97] it was proved that cut-free proofs are cycle-free and it was sketched there that proofs containing cuts but no contractions are also cycle-free.

In this paper we will be concerned with *oriented* cycles but we should notice however that *non-oriented* cycles might appear in proofs and even in cut-free ones. Take for instance any cut-free proof of the pigeon hole principle PHP_n (i.e. the pigeon hole principle relative to n holes: if $n+1$ pigeons are placed in one of n holes, then there is a hole which contains at least two pigeons) written as a sequent $\Gamma_n \rightarrow \Delta_n$ where Γ_n is the collection of formulas

$$\Gamma_n = \{\bigvee_{j=1}^{n} p_{i,j} \;:\; i = 1, \dots, n+1\}$$

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