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A POLYNOMIAL TRANSLATION OF $S4$ INTO INTUITIONISTIC LOGIC

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§1. Introduction. It is known that both $S4$ and the Intuitionistic propositional calculus *Int* are *P-SPACE* complete. This guarantees that there is a polynomial translation from each system into the other.

However, no sound and faithful polynomial translation from $S4$ into *Int* is commonly known. The problem of finding one was suggested by Dana Scott during a very informal gathering of logicians in February 2005 at UCLA. Grigori Mints then brought it to my attention, and in this paper I present a solution. It is based on Kripke semantics and describes model-checking for $S4$ using formulas of *Int*.

A simple translation from *Int* into $S4$, the Gödel-Tarski translation, is well-known; given a formula φ of *Int*, one obtains $\varphi^\Box$ by prefixing $\Box$ to every subformula. For example,

$$(p \vee \neg p)^\Box = \Box(\Box p \vee \Box\neg\Box p).$$

That the translation is sound and faithful can be seen by considering topological semantics, which assign open sets both to $\Box$-formulas of $S4$ and arbitrary formulas of *Int*; the interpretation of φ and $\varphi^\Box$ turn out to be identical. See Tarski's paper [6] for details. Gödel's original paper can be found in [3].

In [2], Friedman and Flagg present a kind of inverse to Gödel-Tarski. Given a formula φ of $S4$ and a finite set of formulas Γ of *Int*, for each $\varepsilon \in \Gamma$ one gets an intuitionistic formula $\varphi_\Gamma^{(\varepsilon)}$ given recursively by

$$\begin{aligned}
\alpha_\Gamma^{(\varepsilon)} &= (\alpha \to \varepsilon) \to \varepsilon \quad \text{for } \alpha \text{ atomic;}\\
(\alpha \vee \beta)_\Gamma^{(\varepsilon)} &= (\alpha_\Gamma^{(\varepsilon)} \vee \beta_\Gamma^{(\varepsilon)} \to \varepsilon) \to \varepsilon;\\
(\alpha \wedge \beta)_\Gamma^{(\varepsilon)} &= \alpha_\Gamma^{(\varepsilon)} \wedge \beta_\Gamma^{(\varepsilon)};\\
(\alpha \to \beta)_\Gamma^{(\varepsilon)} &= \alpha_\Gamma^{(\varepsilon)} \to \beta_\Gamma^{(\varepsilon)};\\
(\Box\alpha)_\Gamma^{(\varepsilon)} &= \Big(\bigwedge_{\gamma\in\Gamma} \alpha_\Gamma^{(\gamma)} \to \varepsilon\Big) \to \varepsilon.
\end{aligned}$$

Usually $\varphi = \psi^\Box$ for some ψ and $\Gamma = \text{sub}(\psi)$. This translation is sound independently of the choice of ε and Γ, but is not faithful in general. For example, if $\varepsilon \equiv \bot \to \bot$ and p is a propositional variable, then

$$p_\Gamma^{(\varepsilon)} = (p \to \varepsilon) \to \varepsilon,$$

which is clearly provable.

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