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CORRESPONDENCES BETWEEN GENTZEN AND HILBERT SYSTEMS

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§1. Introduction. Most Gentzen systems arising in logic contain few axiom schemata and many rule schemata. Hilbert systems, on the other hand, usually contain few proper inference rules and possibly many axioms. Because of this, the two notions tend to serve different purposes. It is common for a logic to be specified in the first instance by means of a Gentzen calculus, whereupon a Hilbert-style presentation 'for' the logic may be sought—or vice versa. Where this has occurred, the word 'for' has taken on several different meanings, partly because the Gentzen separator $\Rightarrow$ can be interpreted intuitively in a number of ways. Here $\Rightarrow$ will be denoted less evocatively by $\rhd$.

In this paper we aim to discuss some of the useful ways in which Gentzen and Hilbert systems may correspond to each other. Actually, we shall be concerned with the *deducibility relations* of the formal systems, as it is these that are susceptible to transformation in useful ways. To avoid potential confusion, we shall speak of Hilbert and Gentzen *relations*. By a *Hilbert relation* we mean any substitution-invariant consequence relation on *formulas*—this comes to the same thing as the deducibility relation of a set of Hilbert-style axioms and rules. By a *Gentzen relation* we mean the fully fledged generalization of this notion in which *sequents* take the place of single formulas. In the literature, Hilbert relations are often referred to as *sentential logics*. Gentzen relations as defined here are their exact *sequential* counterparts. We regard them as logical systems in their own right, rather than as calculi formalizing sentential logics.

Note that we may view any Hilbert relation as a special Gentzen relation by identifying its formulas γ with sequents of the form $\emptyset \rhd \gamma$ and disallowing all other sequent shapes. In the same way, every Gentzen relation that treats sequents of the form $\emptyset \rhd \gamma$ has a *Hilbert subrelation*.

A Gentzen relation can be considered as a complete lattice of *theories*, acted on by a monoid of *substitutions*. In the late 1990s, Jónsson [6, Lec. 1] proposed an algebraically natural notion of *equivalence* between closed set systems with a monoid action. The recently published joint paper [7] of Blok and Jónsson, which subsumes the relevant parts of [6], uses this framework to account in abstract terms for aspects of the correspondence between algebraizable *Hilbert* relations and classes of algebras. The concrete form of this correspondence was described in the

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