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ON A CONJECTURE OF DOBRINEN AND SIMPSON CONCERNING ALMOST EVERYWHERE DOMINATION

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§1. Introduction. Dobrinen and Simpson [4] introduced the notions of almost everywhere domination and uniform almost everywhere domination to study recursion theoretic analogues of results in set theory concerning domination in generic extensions of transitive models of ZFC and to study regularity properties of the Lebesgue measure on 2^ω in reverse mathematics. In this article, we examine one of their conjectures concerning these notions.

Throughout this article, $\leq_T$ denotes Turing reducibility and μ denotes the Lebesgue (or "fair coin") probability measure on 2^ω given by

$$\mu(\{X \in 2^\omega \mid X(n) = i\}) = 1/2.$$

A property holds *almost everywhere* or *for almost all* $X \in 2^\omega$ if it holds on a set of measure 1. For $f, g \in \omega^\omega$, f *dominates* g if $\exists m \forall n > m(f(n) > g(n))$.

DEFINITION 1.1 (Dobrinen, Simpson). A set $A \in 2^\omega$ is *almost everywhere (a.e.) dominating* if for almost all $X \in 2^\omega$ and all functions $g \leq_T X$, there is a function $f \leq_T A$ such that f dominates g. A is *uniformly almost everywhere (u.a.e.) dominating* if there is a function $f \leq_T A$ such that for almost all $X \in 2^\omega$ and all functions $g \leq_T X$, f dominates g.

There are several trivial but useful observations to make about these definitions. First, although these properties are stated for sets, they are also properties of Turing degrees. That is, a set is (u.)a.e. dominating if and only if every other set of the same degree is (u.)a.e. dominating. Second, both properties are closed upwards in the Turing degrees. Third, u.a.e. domination implies a.e. domination. Finally, if A is u.a.e. dominating, then there is a function $f \leq_T A$ which dominates every computable function.

Dobrinen and Simpson [4] introduced these notions to study the following two regularity properties of μ in reverse mathematics: for each G_δ set $Q \subseteq 2^\omega$ and each $\varepsilon > 0$, there is a closed set $F \subseteq Q$ such that $\mu(F) \geq \mu(Q) - \varepsilon$, and for each G_δ set $Q \subseteq 2^\omega$, there is an F_σ set $S \subseteq Q$ such that $\mu(Q) = \mu(S)$. ACA_0 is strong enough to prove these statements, so as the first step toward establishing reversals, they proved the following two theorems. (Reverse mathematics plays only a motivational role here, but the reader who is not familiar with this subject is referred to Simpson [18].)

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