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PFA IMPLIES $\mathsf{AD}^{L(\mathbb{R})}$

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In this paper we shall prove

THEOREM 0.1. *Suppose there is a singular strong limit cardinal κ such that $\square_\kappa$ fails; then AD holds in $L(R)$.*

See [10] for a discussion of the background to this problem. We suspect that more work will produce a proof of the theorem with its hypothesis that κ is a strong limit weakened to $\forall\alpha < \kappa(\alpha^\omega < \kappa)$, and significantly more work will enable one to drop the hypothesis that κ is a strong limit entirely. At present, we do not see how to carry out even the less ambitious project.

Todorcevic [23] has shown that if the Proper Forcing Axiom (PFA) holds, then $\square_\kappa$ fails for all uncountable cardinals κ. Thus we get immediately:

COROLLARY 0.2. PFA *implies* $\mathsf{AD}^{L(\mathbb{R})}$.

It has been known since the early 90's that PFA implies PD, that PFA plus the existence of a strongly inaccessible cardinal implies $\mathsf{AD}^{L(\mathbb{R})}$, and that PFA plus a measurable yields an inner model of $\mathsf{AD}_\mathbb{R}$ containing all reals and ordinals.[1] As we do here, these arguments made use of Tororcevic's work, so that logical strength is ultimately coming from a failure of covering for some appropriate core models.

In late 2000, A. S. Zoble and the author showed that (certain consequences of) Todorcevic's Strong Reflection Principle (SRP) imply $\mathsf{AD}^{L(\mathbb{R})}$. (See [22].) Since Martin's Maximum implies SRP, this gave the first derivation of $\mathsf{AD}^{L(\mathbb{R})}$ from an "unaugmented" forcing axiom.[2]

It should be possible to adapt the techniques of Ketchersid's thesis [2], and thereby strengthen the conclusions of 0.1 and 0.2 to: there is an inner model of AD^+ plus $\Theta_0 < \Theta$ which contains all reals and ordinals. Unpublished work of Woodin shows that the existence of such an inner model implies the existence of a nontame mouse.[3] At the moment, the author sees how to adapt the work in chapter 4 and section 5.1 of [2], but the proof of "branch condensation" in section 5.2 of [2] does not adapt to our situation in any straightforward way. This is the point at which Ketchersid brings in some additional properties of his generic embedding (mainly,

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[1]The first result is due to Woodin, relying heavily on Schimmerling's proof of Δ^1_2 determinacy from PFA. The second result is due to Woodin. For the third, see [1].

[2]In contrast to the arguments referred to in the last paragraph, [22] obtains logical strength from the generic elementary embedding given by a saturated ideal on ω_1, together with simultaneous stationary reflection at ω_2; its argument traces back to Woodin's [24].

[3]The main part of Woodin's proof is described in [17].

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