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TREE STRUCTURES ASSOCIATED TO A FAMILY OF FUNCTIONS

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The research presented in this paper was motivated by our aim to study a problem due to J. Bourgain [3]. The problem in question concerns the uniform boundedness of the classical separation rank of the elements of a separable compact set of the first Baire class. In the sequel we shall refer to these sets (separable or non-separable) as Rosenthal compacta and we shall denote by $\alpha(f)$ the separation rank of a real-valued function f in $\mathscr{B}_1(X)$, with X a Polish space. Notice that in [3], Bourgain has provided a positive answer to this problem in the case of $\mathscr{K}$ satisfying $\mathscr{K} = \overline{\mathscr{K} \cap C(X)}^p$ with X a compact metric space. The key ingredient in Bourgain's approach is that whenever a sequence of continuous functions pointwise converges to a function f, then the possible discontinuities of the limit function reflect a local ℓ^1-structure to the sequence $(f_n)_n$. More precisely the complexity of this ℓ^1-structure increases as the complexity of the discontinuities of f does. This fruitful idea was extensively studied by several authors (c.f. [5], [7], [8]) and for an exposition of the related results we refer to [1]. It is worth mentioning that A.S. Kechris and A. Louveau have invented the rank $r_{ND}(f)$ which permits the link between the c_0-structure of a sequence $(f_n)_n$ of uniformly bounded continuous functions and the discontinuities of its pointwise limit. Rosenthal's c_0-theorem [11] and the c_0-index theorem [2] are consequences of this interaction.

Passing to the case where either $(f_n)_n$ are not continuous or X is a non-compact Polish space, this nice interaction is completely lost. Easy examples show that there exist sequences of continuous functions on $\mathbb{R}$ pointwise convergent to zero and in the same time they are equivalent to the ℓ^1 basis. Also there are sequences $(f_n)_n$ of Baire-1 functions, equivalent to the summing basis of c_0, pointwise convergent to a Baire-2 function. Thus if we wish to preserve the main scheme, invented by Bourgain, namely to pass from the elements of the separable Rosenthal compactum to a well-founded tree related to the dense sequence $(f_n)_n$, this has to take into account not only the finite subsets of $(f_n)_n$ but also the points of the Polish space X. This is the key observation on which we have based our approach. Thus for every $\mathscr{D}$ subset of $\mathbb{R}^X$ we associate a tree $\mathscr{T}\left((f_\xi)_{\xi<\theta}, a, b\right)$ where $(f_\xi)_{\xi<\theta}$ is a well-ordering of $\mathscr{D}$ and $a < b$ are reals. The elements of the tree are of the form (u, T) with u a finite increasing subsequence of $(f_\xi)_{\xi<\theta}$ and T a finite dyadic tree in X, where the length of u and the height of T are the same and which share certain properties.

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