The research presented in this paper was motivated by our aim to study a problem due to J. Bourgain [3]. The problem in question concerns the uniform boundedness of the classical separation rank of the elements of a separable compact set of the first Baire class. In the sequel we shall refer to these sets (separable or non-separable) as Rosenthal compacta and we shall denote by $\alpha(f)$ the separation rank of a real-valued function $f$ in $B_1(X)$, with $X$ a Polish space. Notice that in [3], Bourgain has provided a positive answer to this problem in the case of $K$ satisfying $K = K \setminus C(X)$ with $X$ a compact metric space. The key ingredient in Bourgain’s approach is that whenever a sequence of continuous functions pointwise converges to a function $f$, then the possible discontinuities of the limit function reflect a local $\ell^1$-structure to the sequence $(f_n)_n$. More precisely the complexity of this $\ell^1$-structure increases as the complexity of the discontinuities of $f$ does.

This fruitful idea was extensively studied by several authors (c.f. [5], [7], [8]) and for an exposition of the related results we refer to [1]. It is worth mentioning that A.S. Kechris and A. Louveau have invented the rank $r_{ND}(f)$ which permits the link between the $c_0$-structure of a sequence $(f_n)_n$ of uniformly bounded continuous functions and the discontinuities of its pointwise limit. Rosenthal’s $c_0$-theorem [11] and the $c_0$-index theorem [2] are consequences of this interaction.

Passing to the case where either $(f_n)_n$ are not continuous or $X$ is a non-compact Polish space, this nice interaction is completely lost. Easy examples show that there exist sequences of continuous functions on $\mathbb{R}$ pointwise convergent to zero and in the same time they are equivalent to the $\ell^1$ basis. Also there are sequences $(f_n)_n$ of Baire-1 functions, equivalent to the summing basis of $c_0$. pointwise convergent to a Baire-2 function. Thus if we wish to preserve the main scheme, invented by Bourgain, namely to pass from the elements of the separable Rosenthal compactum to a well-founded tree related to the dense sequence $(f_n)_n$, this has to take into account not only the finite subsets of $(f_n)_n$ but also the points of the Polish space $X$.

This is the key observation on which we have based our approach. Thus for every $D$ subset of $\mathbb{R}^X$ we associate a tree $\mathcal{T}((f_z)_{z \in D}, a, b)$ where $(f_z)_{z \in D}$ is a well-ordering of $D$ and $a < b$ are reals. The elements of the tree are of the form $(u, T)$ with $u$ a finite increasing subsequence of $(f_z)_{z \in D}$ and $T$ a finite dyadic tree in $X$, where the length of $u$ and the height of $T$ are the same and which share certain properties.