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THEORIES OF PRESHEAF TYPE

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Introduction. Let us say that a geometric theory T is of *presheaf type* if its classifying topos $\mathbb{B}[T]$ is (equivalent to) a presheaf topos. (We adhere to the convention that *geometric logic* allows arbitrary disjunctions, while *coherent logic* means geometric and finitary.) Write $\mathrm{Mod}(T)$ for the category of *Set*-models and homomorphisms of T. The next proposition is well known; see, for example, MacLane–Moerdijk [13], pp. 381-386, and the textbook of Adámek–Rosický [1] for additional information:

PROPOSITION 0.1. *For a category $\mathscr{M}$, the following properties are equivalent*:

(i) *$\mathscr{M}$ is a finitely accessible category in the sense of Makkai–Paré* [14], *i.e., it has filtered colimits and a small dense subcategory $\mathscr{C}$ of finitely presentable objects*
(ii) *$\mathscr{M}$ is equivalent to* $\mathrm{Pts}(Set^{\mathscr{C}})$, *the category of points of some presheaf topos*
(iii) *$\mathscr{M}$ is equivalent to the free filtered cocompletion* (*also known as Ind-$\mathscr{C}$*) *of a small category $\mathscr{C}$.*
(iv) *$\mathscr{M}$ is equivalent to* $\mathrm{Mod}(T)$ *for some geometric theory of presheaf type.*

Moreover, if these are satisfied for a given $\mathscr{M}$, then the $\mathscr{C}$—in any of (i), (ii) *and* (iii)*—can be taken to be the full subcategory of $\mathscr{M}$ consisting of finitely presentable objects.* (*There may be inequivalent choices of $\mathscr{C}$, as it is in general only determined up to idempotent completion*; *this will not concern us.*)

This seems to completely solve the problem of identifying when T is of presheaf type: check whether $\mathrm{Mod}(T)$ is finitely accessible and if so, recover the presheaf topos as *Set*-functors on the full subcategory of finitely presentable models. There is a subtlety here, however, as pointed out (probably for the first time) by Johnstone [10]. It is exemplified by the word *some* in (iv) above. Namely, the presheaf topos one recovers this way (which indeed has $\mathscr{M}$ as its category of *Set*-models) need not coincide with the sought-for topos $\mathbb{B}[T]$. Take, for example, any axiomatization T_1 of the theory of fields by coherent sentences. (We take this merely to mean that $\mathrm{Mod}(T_1)$ is equivalent to the category of fields and homomorphisms.) That category is finitely accessible, so there are geometric theories T_2 of presheaf type such that $\mathrm{Mod}(T_2)$ is the category of fields. But T_1 is not one of them; there exists no coherent presheaf type axiomatization of fields. (See Cor. 2.2 below.) Such a T_1

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