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DIOPHANTINE PROPERTIES OF SETS DEFINABLE IN O-MINIMAL STRUCTURES

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§1. Introduction. Let $\mathbb{M}$ be an o-minimal expansion of the ordered field of real numbers $\overline{\mathbb{R}}$, and let S be an $\mathbb{M}$-definable subset (parameters allowed unless otherwise stated) of $\mathbb{R}^n$. In this note I investigate questions concerning the distribution of points on S with *integer* coordinates. My main theorem gives an estimate which, though probably far from best possible, at least shows that the o-minimal assumption does have diophantine consequences. This is, perhaps, surprising in view of the flexibility that we now seem to have in constructing o-minimal expansions of $\overline{\mathbb{R}}$ (see, e. g. [7], [8], [9]).

1.1. MAIN THEOREM. *With $\mathbb{M}$, S as above, suppose further that S is of dimension* 1 *and that it contains no unbounded $\overline{\mathbb{R}}$-definable* (*without parameters*) *subset. Let $\varepsilon > 0$ be given. Then for all sufficiently large R, the set $S \cap \mathbb{Z}^n$ contains at most R^ε points within any* (*euclidean*) *ball in $\mathbb{R}^n$ of diameter R.*

The restriction that S be eventually transcendental is obviously necessary (e.g., consider the graph of any univariate polynomial with integer coefficients), but it would be interesting to weaken the hypothesis that $\dim(S) = 1$. I should also remark that for most o-minimal structures known to me, the bound R^ε can be improved to a power of $\log R$ and this is, I would guess, the right order of magnitude in general. One certainly cannot do better, as the $\langle \mathbb{R}, \exp \rangle$-definable set $\{\langle x, y \rangle \in \mathbb{R}^2 : x = 2^y\}$ shows. (Or, for a polynomially bounded example, consider the expansion of $\overline{\mathbb{R}}$ by the set $\{\langle x, y \rangle \in \mathbb{R}^2 : x = y^\alpha\}$ where, say, $\alpha = \log_2 3$.) However, in one interesting case, namely that of $\mathbb{M} = \mathbb{R}_{an}$ (where the definable sets are exactly the globally subanalytic sets — see [2]), one can show that $\log \log R$ is the correct order of magnitude in the sense that if S is as in the Main Theorem, then there exists a constant $C > 0$ such that for all sufficiently large R the set $S \cap \mathbb{Z}^n$ contains at most $C \log \log R$ points within any ball in $\mathbb{R}^n$ of diameter R. This follows easily from my second result:–

1.2. THEOREM. *Let $\langle a_n : n \geq 1 \rangle$ be a strictly increasing sequence of positive integers. Then the following are equivalent*:

(a) *there exists an $\mathbb{R}_{an}$-definable function $f : \mathbb{R} \to \mathbb{R}$ such that $f(a_n) \in \mathbb{Z}$ for all $n \geq 1$, and such that for no $r \in \mathbb{R}$ is $f \upharpoonright (r, \infty)$ an $\overline{\mathbb{R}}$-definable function* (*without parameters*);

(b) *there exists an integer $N \geq 1$ such that for all positive integers n, $a_{n+N} \geq a_n^2$.*

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