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REVERSE MATHEMATICS AND THE EQUIVALENCE OF DEFINITIONS FOR WELL AND BETTER QUASI-ORDERS

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§1. Introduction. In reverse mathematics, one formalizes theorems of ordinary mathematics in second order arithmetic and attempts to discover which set theoretic axioms are required to prove these theorems. Often, this project involves making choices between classically equivalent definitions for the relevant mathematical concepts. In this paper, we consider a number of equivalent definitions for the notions of well quasi-order and better quasi-order and examine how difficult it is to prove the equivalences of these definitions.

As usual in reverse mathematics, we work in the context of subsystems of second order arithmetic and take RCA_0 as our base system. RCA_0 is the subsystem formed by restricting the comprehension scheme in second order arithmetic to Δ^0_1 formulas and adding a formula induction scheme for Σ^0_1 formulas. For the purposes of this paper, we will be concerned with fairly weak extensions of RCA_0 (indeed strictly weaker than the subsystem ACA_0 which is formed by extending the comprehension scheme in RCA_0 to cover all arithmetic formulas) obtained by adjoining certain combinatorial principles to RCA_0. Among these, the most widely used in reverse mathematics is Weak König's Lemma; the resulting theory WKL_0 is extensively documented in [11] and elsewhere.

We give three other combinatorial principles which we use in this paper. In these principles, we use k to denote not only a natural number but also the finite set $\{0, \dots, k-1\}$. For any set X and any $n \in \mathbb{N}$, we let $[X]^n$ denote the set of all subsets of X of size n. Similarly, $[X]^{<\omega}$ denotes the set of all finite subsets of X and $Y \in [X]^\omega$ is an abbreviation for the statement that Y is an infinite subset of X. The Pigeonhole principle is the statement

$$\forall k\, \forall f : \mathbb{N} \to k\, \exists A \in [\mathbb{N}]^\omega\ \exists i < k\, \forall j \in A\, (f(j) = i)$$

and is denoted $\mathsf{RT}^1_{<\infty}$. (This notation comes from thinking of the principle as a version of Ramsey's Theorem for singletons and finitely many colors.) Ramsey's Theorem for pairs and two colors is the statement

$$\forall f : [\mathbb{N}]^2 \to 2\, \exists A \in [\mathbb{N}]^\omega\ \exists i < 2\, \forall j \in [A]^2\, (f(j) = i)$$

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