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Π^1_1 RELATIONS AND PATHS THROUGH $\mathcal{O}$

SERGEY S. GONCHAROV, VALENTINA S. HARIZANOV, JULIA F. KNIGHT, AND RICHARD A. SHORE

§1. Introduction. When bounds on complexity of some aspect of a structure are preserved under isomorphism, we refer to them as *intrinsic*. Here, building on work of Soskov [34], [33], we give syntactical conditions necessary and sufficient for a relation to be intrinsically Π^1_1 on a structure. We consider some examples of computable structures $\mathcal{A}$ and intrinsically Π^1_1 relations R. We also consider a general family of examples of intrinsically Π^1_1 relations arising in computable structures of maximum Scott rank.

For three of the examples, the maximal well-ordered initial segment in a Harrison ordering, the superatomic part of a Harrison Boolean algebra, and the height-possessing part of a Harrison p-group, we show that the Turing degrees of images of the relation in computable copies of the structure are the same as the Turing degrees of Π^1_1 paths through Kleene's $\mathcal{O}$. With this as motivation, we investigate the possible degrees of these paths. We show that there is a Π^1_1 path in which $\emptyset'$ is not computable. In fact, there is one in which no noncomputable hyperarithmetical set is computable.[1] There are paths that are Turing incomparable, or Turing incomparable over a given hyperarithmetical set. There is a pair of paths whose degrees form a minimal pair. However, there is no path of minimal degree.

In Section 2, we summarize earlier results on intrinsically c.e. and intrinsically Σ^0_α relations. In Section 3, we rework Soskov's results, and we give our result on intrinsically Π^1_1 relations. In Section 4, we describe the examples. In Section 5, we show that for the well-ordered initial segment of the Harrison ordering and related examples, the degrees of images of the relation in computable copies of the structure match those of Π^1_1 paths through $\mathcal{O}$. In Section 6, we give results on degrees of paths through $\mathcal{O}$. In the remainder of the present section, we give some background. Most of this material may be found in the book by Ash and Knight [3].

1.1. Kleene's $\mathcal{O}$. We give a brief description of Kleene's system of notation for computable ordinals. Further details may be found in [29] or [3]. The system consists of a set $\mathcal{O}$ of notations, together with a partial ordering $<_{\mathcal{O}}$. The ordinal 0 gets notation 1. If a is a notation for α, then 2^a is a notation for $\alpha + 1$. Then $a <_{\mathcal{O}} 2^a$, and also, if $b <_{\mathcal{O}} a$, then $b <_{\mathcal{O}} 2^a$. Suppose α is a limit ordinal. If φ_e is

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[1] This provides a new solution to Problem 71 on H. Friedman's list [10].

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