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ON A QUESTION OF HERZOG AND ROTHMALER

ANAND PILLAY AND MARTIN ZIEGLER†

§1. Introduction. Herzog and Rothmaler gave the following purely topological characterization of stable theories. (See the exercises 11.3.4 – 11.3.7 in [2]).
A complete theory T is stable iff for any model M and any extension $M \subset B$ the restriction map $\mathrm{S}(B) \to \mathrm{S}(M)$ has a continuous section.

In fact, if T is stable, taking the unique non-forking extension defines a continuous section of $\mathrm{S}(B) \to \mathrm{S}(A)$ for all subsets A of B, provided A is algebraically closed in T^{eq}. Herzog and Rothmaler asked, if, for stable T, there is a continuous section for *any* subset A of B. Or, equivalently, if for any A, $\mathrm{S}(\mathrm{acl}^{\mathrm{eq}}(A)) \to \mathrm{S}(A)$ has a continuous section.

This is an interesting problem, also for unstable T. Is it true that for any T and any set of parameters A the restriction map $\mathrm{S}(\mathrm{acl}(A)) \to \mathrm{S}(A)$ has a continuous section? We answer the question by the following two theorems.

THEOREM 1. *Let A be a subset of a model of T. Assume that the Boolean algebra of $\mathrm{acl}(A)$–definable formulas is generated by*

- *some countable set of formulas,*
- *all A–definable formulas,*
- *all formulas which are atomic over $\mathrm{acl}(A)$.*

Then $\mathrm{S}(\mathrm{acl}(A)) \to \mathrm{S}(A)$ has a continuous section.

The conditions of the theorems are satisfied if, for example, L and A are countable, or, if there are only countably many non–isolated types over $\mathrm{acl}(A)$.

THEOREM 2. *There is a theory of Morley rank 2 and Morley degree 1 such that $\mathrm{S}(\mathrm{acl}(\emptyset)) \to \mathrm{S}(\emptyset)$ has no continuous section.*

§2. Proof of Theorem 1. Theorem 1 follows immediately from the next lemma. (Note that the map $\mathrm{S}(\mathrm{acl}(A)) \to \mathrm{S}(A)$ is always open).

LEMMA 3. *Let A be a subalgebra of the Boolean algebra B such that the projection of Stone spaces $\mathrm{S}(B) \to \mathrm{S}(A)$ is open. Assume that B can be generated by*

- *some countable subalgebra C of B,*
- *the elements of A,*
- *all atoms of B.*

Then the projection has a continuous section $\sigma : \mathrm{S}(A) \to \mathrm{S}(B)$.

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