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A PARTIAL ANALYSIS OF MODIFIED REALIZABILITY

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§1. Introduction. A formalized version of Kleene realizability for intuitionistic first-order arithmetic **HA** was axiomatically characterized by Troelstra (see [2, 3.2]) as follows: for an arbitrary **HA**-sentence φ, $\mathbf{HA} \vdash \exists x(x \text{ realizes } \varphi)$ if and only if $\mathbf{HA} + \mathrm{ECT}_0 \vdash \varphi$.

Many notions of realizability have been characterized in this fashion: see [2] or [3] for details. For some notions, for example extensional realizability, it is necessary to pass to an extension of **HA**: realizability in **HA** is characterized by deducibility from certain axioms in an extension of **HA**.

The present note is concerned with *modified realizability*, seen as interpretation for **HA**. From semantical considerations (see [4]) it follows that this interpretation can be constructed as a combination of three ingredients:

i) Kleene realizability;
ii) Kripke forcing over a 2-element linear order P;
iii) The Friedman translation [1].

This will be shown in section 2. The Friedman translation (in the way we use it) introduces a new propositional constant V; hence we move to an extension $\mathbf{HA}(V)$ of **HA**. We must then define Kleene realizability and forcing for the extended language. Now let $(\varphi)_V$ be the result of the Friedman translation applied to φ. We obtain, in **HA**, that the sentence saying that φ is modified-realizable, is equivalent to the sentence which says that the statement "$(\varphi)_V$ is Kleene-realizable" is forced in P (see section 2).

Therefore, the following programme suggests itself: since we know how to characterize Kleene realizability, if we can characterize also the forcing interpretation, we might be able to put these characterizations together in order to obtain our desired result.

However, in this paper we see that this straightforward approach runs into an obstacle, in a way the author thinks is surprising. A more refined analysis yields a partial result: instead of an equivalence, we only obtain one implication. We formulate a conjecture under which the partial result does give a complete characterization.

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