THE JOURNAL OF SYMBOLIC LOGIC
Volume 69, Number 2, June 2004

ON FIRST-ORDER SENTENCES WITHOUT FINITE MODELS

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We will mainly be concerned with a result which refutes a stronger variant of a conjecture of Macpherson about finitely axiomatizable ω-categorical theories. Then we prove a result which implies that the ω-categorical stable pseudoplanes of Hrushovski do not have the finite submodel property.

Let's call a consistent first-order sentence without finite models an *axiom of infinity*. Can we somehow describe the axioms of infinity? Two standard examples are:
ϕ_1 : A first-order sentence which expresses that a binary relation $<$ on a nonempty universe is transitive and irreflexive and that for every x there is y such that $x < y$.
ϕ_2 : A first-order sentence which expresses that there is a unique x such that,

(0) for every y, $s(y) \neq x$ (where s is a unary function symbol),

and, for every x, if x does not satisfy (0) then there is a unique y such that $s(y) = x$.

Every complete theory T such that $\phi_1 \in T$ has the strict order property (as defined in [10]), since the formula $x < y$ will have the strict order property for T. Let's say that if ψ is an axiom of infinity and every complete theory T with $\psi \in T$ has the strict order property, then ψ *has the strict order property*.

Every complete theory T such that $\phi_2 \in T$ is not ω-categorical. This is the case because a complete theory T without finite models is ω-categorical if and only if, for every $0 < n < \omega$, there are only finitely many formulas in the variables $x_1, \ldots, x_n$, up to equivalence, in any model of T. It is easy to see that all the formulas

$$s(x_1) = x_2,\ s(s(x_1)) = x_2,\ s(s(s(x_1))) = x_2,\ \ldots$$

are mutually nonequivalent in any model of ϕ_2. Let's say that if ψ is an axiom of infinity and every complete theory T with $\psi \in T$ is not ω-categorical, then ψ *refutes ω-categoricity*.

The question arises whether every axiom of infinity has the strict order property or refutes ω-categoricity. This question is related to a conjecture of Macpherson [9], saying: Every finitely axiomatized ω-categorical theory with infinite models has the strict order property. We may assume that the conjecture speaks only about complete theories because otherwise the finitely axiomatized theory expressing the axioms of a vector space over the field with two elements is a counterexample (since it is ω-categorical and every infinite model of it is stable). Let's say that an axiom of infinity is *complete* if it axiomatizes a complete theory. Now Macpherson's conjecture can be rephrased as: Every complete axiom of infinity refutes ω-categoricity or has the

Received May 22, 2003; revised September 15, 2003.

0022-4812/04/6902-0001/$2.10