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MODEL THEORY OF COMODULES

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The purpose of this paper is to establish some basic points in the model theory of comodules over a coalgebra. It is not even immediately apparent that there is a model theory of comodules since these are not structures in the usual sense of model theory. Let us give the definitions right away so that the reader can see what we mean.

Fix a field k. A k-*coalgebra* C is a k-vector space equipped with a k-linear map $\Delta\colon C \longrightarrow C \otimes C$, called the *comultiplication* (by $\otimes$ we always mean tensor product over k), and a k-linear map $\varepsilon\colon C \longrightarrow k$, called the *counit*, such that $\Delta \otimes 1_C = 1_C \otimes \Delta$ (coassociativity) and $(1_C \otimes \varepsilon)\Delta = 1_C = (\varepsilon \otimes 1_C)\Delta$, where we identify C with both $k \otimes C$ and $C \otimes k$. These definitions are literally the duals of those for a k-algebra: express the axioms for C' to be a k-algebra in terms of the multiplication map $\mu\colon C' \otimes C' \longrightarrow C'$ and the "unit" (embedding of k into C'), $\delta\colon k \longrightarrow C'$ in the form that certain diagrams commute and then just turn round all the arrows. See [5] or more recent references such as [7] for more.

A (right) *comodule* over the coalgebra C is a k-vector space M equipped with a k-linear map $\rho\colon M \longrightarrow M \otimes C$ which satisfies $1_M \otimes \Delta = \rho \otimes 1_C$ and $(1_M \otimes \varepsilon)\rho = 1_M$, where we identify M and $M \otimes k$ (and, of course, $M \otimes (C \otimes C)$ with $(M \otimes C) \otimes C$). Again, the way to understand this definition is to write the axioms for being a unital module M' over an algebra C' in terms of the structure map $M' \otimes C' \longrightarrow M$ in a diagrammatic way and then reverse all arrows.

The structure on a C-comodule is, therefore, the structure of a k-vector space (which is no problem) together with a morphism from M to $M \otimes C$. Recall that what we do with the structure map $M \otimes C' \longrightarrow M'$ of a module M' is to build each function $- \otimes c\colon M' \longrightarrow M'$, for $c \in C'$, into the language. It is not so clear how to proceed in the case of comodules. That is, does there exist a language in which one may axiomatise the concept of a C-comodule, where C is a fixed k-coalgebra?

It is not difficult to give plausible reasons as to why this question should have a negative answer. But plausibility is not enough, as we shall see.

A key fact that we use is the equivalence of the category of C-comodules with a subcategory of the category of C^*-modules, where C^* is the *dual algebra* of C. As a vector space, C^* is the dual, $\mathrm{Hom}_k(C, k)$, of k and it is easy to verify that the k-coalgebra structure of C induces, in a natural way, the structure of a k-algebra on C^* (if $f, g \in C^*$, $c \in C$ with $\Delta c = \sum_i c_i' \otimes c_i''$, set $(fg)c = \sum_i f(c_i')g(c_i'')$).

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