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VALUATION THEORETIC CONTENT OF THE MARKER-STEINHORN THEOREM

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The Marker-Steinhorn Theorem (cf. [2] and [3]), says the following. If T is an o-minimal theory and $M \prec N$ is an elementary extension of models of T such that M is Dedekind complete in N, then for every N-definable subset X of N^k, the trace $X \cap M^k$ is M-definable. The original proof in [2] gives an explicit method how to construct a defining formula of $X \cap M^k$ out of a defining formula of X. A geometric reformulation of the Marker-Steinhorn Theorem is the definability of Hausdorff limits of families of definable sets. An explicit construction of these Hausdorff limits for expansions of the real field has recently been achieved in [1]. Both proofs and also the treatment [3] are technically involved.

Here we give a short algebraic, but not constructive proof, if T is an expansion of real closed fields. In fact we'll identify the statement of the Theorem with a valuation theoretic property of models of T (namely condition (†) below). Therefore our proof might be applicable to other elementary classes which expand fields, if a notion of dimension and a reasonable valuation theory are available.

From now on, let T be an o-minimal expansion of real closed fields. We have to show the following (cf. [2], Th. 2.1. for this formulation). If M is a model of T and p is a tame n-type over M (i.e., M is Dedekind complete in $M\langle\bar{\alpha}\rangle := \mathrm{dcl}(M\bar{\alpha})$ for some realization $\bar{\alpha}$ of p), then p is a definable type (cf. [4], 11.b).

We fix some $n \in \mathbb{N}$ and prove by induction on k the following:

$(*)_k$ If $M \prec N$ are models of T, with $\dim N/M = k$ and p is a tame n-type of M, then p has a unique heir q on N and q is tame again.

REMARK. In order to prove that a type p is definable it is enough to show that p has a unique heir on N for all $N \succ M$ with $\dim N/M < \infty$, where dim denotes the dimension in the sense of T (cf. [4], Th. 11.07). So the Theorem is proved if we know $(*)_k$ for all k. The additional condition (that q is tame) in $(*)_k$ is needed for the induction step. The induction step is easy and does not use any valuation theory. For $k = 1$ it is obvious that p has a unique heir on N, if p is tame. Therefore the essence of all the $(*)_k$ is the tameness assertion for q in $(*)_1$. We'll rephrase this assertion in the following way:

(†) If $M^* \prec N^*$ are models of T with $\dim N^*/M^* = 1$ and W is a T-convex valuation ring of N^*, then the dimension of the residue field of W over the residue field of $W \cap M^*$ is at most 1.

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