THE JOURNAL OF SYMBOLIC LOGIC
Volume 69, Number 1, March 2004

TWO STEP ITERATION OF ALMOST DISJOINT FAMILIES

JERRY E. VAUGHAN

§1. Introduction. Let E be an infinite set, and $[E]^\omega$ the set of all countably infinite subsets of E. A family $\mathscr{A} \subset [E]^\omega$ is said to be *almost disjoint* (respectively, *pairwise disjoint*) provided for $A, B \in \mathscr{A}$, if $A \neq B$ then $A \cap B$ is finite (respectively, $A \cap B$ is empty). Moreover, an infinite family $\mathscr{A}$ is said to be a *maximal almost disjoint family* provided it is an infinite almost disjoint family not properly contained in any almost disjoint family. In this paper we are concerned with the following set of topological spaces defined from (maximal) almost disjoint families of infinite subsets of the natural numbers ω.

DEFINITION 1.1. For an almost disjoint family $\mathscr{A}_0 \subset [\omega]^\omega$, define a topological space $\psi(\mathscr{A}_0) = \omega \cup \mathscr{A}_0$ with the topology in which for every natural number $n \in \omega$, the singleton set $\{n\}$ is a local base at n (i.e., n is an isolated point), and each $A \in \mathscr{A}_0$ has a local base consisting of sets of the form $\{A\} \cup A \setminus F$ where F is a finite subset of ω.

The space $\psi(\mathscr{A}_0)$ is well known. If the almost disjoint family is denoted by $\mathscr{R}$ and the natural numbers by $\mathscr{N}$ then the space $\psi(\mathscr{A}_0)$ was called $\mathscr{N} \cup \mathscr{R}$ by S. Mrówka [3]. The same space was called Ψ in [1, Exercise 5I], where it is attributed to J. Isbell.

All the results in this paper can be stated in terms of the spaces $\psi(\mathscr{A}_0)$ (see Remark 2.1) but for reasons of convenience and motivation (see §2), we prefer to consider $\psi(\mathscr{A}_0)$ as a subspace of a larger space, which we now define.

DEFINITION 1.2. Given $\psi(\mathscr{A}_0)$, let $\mathscr{A}_1 \subset [\mathscr{A}_0]^\omega$ be a maximal almost disjoint family. Define a topological space $\psi(\mathscr{A}_0, \mathscr{A}_1) = \psi(\mathscr{A}_0) \cup \mathscr{A}_1 = \omega \cup \mathscr{A}_0 \cup \mathscr{A}_1$ with the topology in which local bases for points in $\omega \cup \mathscr{A}_0$ are taken to be the same as in $\psi(\mathscr{A}_0)$, and a local base for a point $X \in \mathscr{A}_1$ consists of all sets of the form

$$\{X\} \cup (X \setminus G) \cup (\cup\{A \setminus F(A) : A \in X \setminus G\})$$

where G is finite, and $F(A)$ is finite for all $A \in X \setminus G$.

The space $\psi(\mathscr{A}_0, \mathscr{A}_1)$, called a *two step iteration of* ψ, was introduced in [6].

Received August 19, 2002; revised August 31, 2003.
2000 *Mathematics Subject Classification.* 03E17, 54D10, Secondary 03E35, 03E75, 54A20, 54G20.
Key words and phrases. almost disjoint families, small uncountable cardinals, iterations of ψ, Hausdorff, Urysohn.

0022-4812/04/6901-0008/$2.00