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PARTITIONING LARGE VECTOR SPACES

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The theme of this paper is the generalization of theorems about partitions of the sets of points and lines of finite-dimensional Euclidean spaces $\mathbb{R}^d$ to vector spaces over $\mathbb{R}$ of arbitrary dimension and, more generally still, to arbitrary vector spaces over other fields so long as these fields are not too big. These theorems have their origins in the following striking theorem of Sierpiński [12] which appeared a half century ago.

SIERPIŃSKI'S THEOREM. *The Continuum Hypothesis is equivalent to: There is a partition $\{X, Y, Z\}$ of $\mathbb{R}^3$ such that if ℓ is a line parallel to the x-axis* [respectively: *y-axis, z-axis*] *then $X \cap \ell$* [respectively: $Y \cap \ell$, $Z \cap \ell$] *is finite.*

The history of this theorem and some of its subsequent developments are discussed in the very interesting article by Simms [13]. Sierpiński's Theorem was generalized by Kuratowski [9] to partitions of $\mathbb{R}^{n+2}$ into $n + 2$ sets obtaining an equivalence with $2^{\aleph_0} \leq \aleph_n$. The geometric character that Sierpiński's Theorem and its generalization by Kuratowski appear to have is bogus, since the lines parallel to coordinate axes are essentially combinatorial, rather than geometric, objects. The following version of Kuratowski's theorem emphasizes its combinatorial character.

KURATOWSKI'S THEOREM. *Let $n < \omega$ and A be any set. Then $|A| \leq \aleph_n$ if and only if there is a partition $P \colon A^{n+2} \longrightarrow n + 2$ such that if $i \leq n + 1$ and ℓ is a line parallel to the i-th coordinate axis, then $\{ x \in \ell : P(x) = i \}$ is finite.*

Davies [2] obtained a further generalization of Sierpiński's Theorem, restoring its geometric character. Let $\mathscr{L}(\mathbb{R}^d)$ be the set of lines in d-dimensional Euclidean space. The concept of a line makes sense not only in $\mathbb{R}^d$ or any other vector space over $\mathbb{R}$, but in any vector space at all. Let $\mathscr{L}(V)$ be the set of lines in any vector space V over a field, where a line is a translate of a 1-dimensional subspace. The following theorem is the prototype of the main new results that are presented in this paper.

DAVIES'S THEOREM. *Assume $n < \omega$ and $2^{\aleph_0} \leq \aleph_n$. Let V be a finite-dimensional vector space over $\mathbb{R}$, and let $L \colon \mathscr{L}(V) \longrightarrow n + 2$. Then there is $P \colon V \longrightarrow n + 2$ such that $\{ x \in \ell : P(x) = L(\ell) \}$ is finite for each $\ell \in \mathscr{L}(V)$.*

It is natural to ask about extending Davies's Theorem to other vector spaces. Corollary 1.2 in §1 of this paper is a result in this direction which applies to arbitrary vector spaces over any small enough field. The main point here is that there is no restriction on $\dim(V)$, just on the cardinality of the field.

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