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ELEMENTARY PROPERTIES OF THE BOOLEAN HULL AND REDUCED QUOTIENT FUNCTORS

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§1. Introduction. In [12] we proved the following isotropy-reflection principle:

THEOREM. *Let F be a formally real field and let F^p denote its Pythagorean closure. The natural embedding of reduced special groups from $G_{\text{red}}(F)$ into $G_{\text{red}}(F^p) = G(F^p)$ induced by the inclusion of fields, reflects isotropy.*

Here $G_{\text{red}}(F)$ denotes the reduced special group (with underlying group $\dot{F}/\Sigma\dot{F}^2$) associated to the field F, henceforth assumed formally real; cf. [11], Chapter 1, §3, for details.

The result proved in [12] is, in fact, more general. For example, the Pythagorean closure F^p can be replaced in the statement above by the intersection of all real closures of F (inside a fixed algebraic closure). Similar statements hold, more generally, for all *relative* Pythagorean closures of F in the sense of Becker [3], Chapter II, §3.

Since the notion of isotropy of a quadratic form can be expressed by a first-order formula in the natural language L_{SG} for special groups (with the coefficients as parameters), this result raises the question whether the embedding $\iota_{FF^p} : G_{\text{red}}(F) \hookrightarrow G(F^p)$ is elementary. Further, since the L_{SG}-formula expressing isotropy is positive-existential, one may also ask whether ι_{FF^p} reflects all (closed) formulas of that kind with parameters in $G_{\text{red}}(F)$.

In this paper we give a negative answer to the first of these questions, for a vast class of formally real (non-Pythagorean) fields F (Prop. 5.1). This follows from rather general preservation results concerning the "Boolean hull" and the "reduced quotient" operations on special groups.

In Chapter 4, Section 2 of [11] we introduced the *Boolean hull* operation which to every reduced special group (RSG) G associates a Boolean algebra (BA), its Boolean hull B_G. The correspondence $G \hookrightarrow B_G$ is functorial. In Section 3 we show that this functor preserves elementary equivalence and embeddings for the following classes:

(1) Reduced special groups whose space of orders have a finite number of isolated points (Corollary 3.7 (c) and Proposition 3.12);
(2) Reduced special groups of finite chain length (Proposition 3.14);
(3) Reduced special groups of finite stability index (Proposition 3.18).

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